

Rationality of vertex operator algebras

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Abstract

It is shown that a simple vertex operator algebra V is rational if and only if its Zhu algebra $A(V)$ is semisimple and each irreducible admissible V -module is ordinary. A contravariant form on a Verma type admissible V -module is constructed and the radical is exactly the maximal proper submodule. As an application the rationality of V_L^+ for any positive definite even lattice is obtained.

1 Introduction

One of the most important problems in the representation theory of vertex operator algebras is to determine various module categories. Although there are several ways to define modules for a vertex operator algebra for different purposes, there are essentially three different notions of modules, namely,

weak modules \supset admissible modules \supset ordinary modules.

The ordinary module was first defined in [FLM] in the construction of moonshine vertex operator algebra V^\natural . The ordinary modules are graded by the eigenvalues of the degree operator $L(0)$ with finite dimensional eigenspaces and the eigenvalues are bounded below. In order to study the modular invariance of trace functions for vertex operator algebras, the admissible module was introduced in [Z] (see also [DLM2]). The admissible modules are \mathbb{Z}_+ -graded but the homogenous spaces are not necessarily finite dimensional and $L(0)$ is not assumed to be semisimple. The weak modules do not have any grading restriction.

There are two basics in understanding module categories. The first one is the classification of irreducible objects and the other is whether or not the category is semisimple. For vertex operator algebras, the admissible module category is the most important one for various considerations. We call a vertex operator algebra *rational* if the admissible

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module category is semisimple [DLM2]. This rationality definition is essentially the rationality first defined in [Z] with two more assumptions: (a) There are only finitely many irreducible admissible modules, (b) each irreducible admissible module is ordinary. It was proved in [DLM2] that (a) and (b) follow from the semisimplicity of the admissible module category. Many well known vertex operator algebras (such as lattice vertex operator algebras [B], [FLM], [D1], [DLM1], the integrable affine vertex operator algebras [FZ], the Virasoro vertex operator algebras associated to the discrete series [W1], [DMZ], the framed and code vertex operator algebras [D2], [DGH], [M], certain W -algebras [FB]) are rational.

Both the classification of irreducible admissible modules and the rationality for a vertex operator algebra V are inseparable with an associative algebra $A(V)$ attached to V . Defined in [Z], the associative algebra $A(V)$ which is a quotient of V plays a fundamental role in the classification of irreducible admissible modules. It is proved in [Z] and [DLM2] that there is a one to one correspondence between the equivalence classes of irreducible admissible V -modules and the equivalence classes of simple $A(V)$ -modules. The correspondence is given by sending the admissible module to its bottom homogeneous subspace as the admissible module is truncated from below. So in some sense, the $A(V)$ controls the bottom level of an admissible module. The relation between admissible modules for a vertex operator algebra and their bottom levels for $A(V)$ can be regarded as an analogue of classical highest weight module theory in the theory of vertex operator algebras. The algebra $A(V)$ is computable for many vertex operator algebras. The classifications of irreducible admissible modules for many well known vertex operator algebras have been achieved using the $A(V)$ -theory. Examples include the affine vertex operator algebras [FZ], the Virasoro vertex operator algebras [DMZ], [W1], lattice type vertex operator algebras [DN1]-[DN3], [AD], [TY], certain W -algebras [W2], [DLTY] and some other vertex operator algebras [A2], [Ad], [KW], [KMY]. So the classification of irreducible admissible modules for a vertex operator algebra, in principle, can be done.

It is proved in [Z] and [DLM2] that if V is rational then $A(V)$ is a finite dimensional semisimple associative algebra. A natural question is

Does the semisimplicity of $A(V)$ imply the rationality of V ?

In this paper we give a positive answer to the question and prove that a simple vertex operator algebra V is rational if and only if $A(V)$ is semisimple and each irreducible admissible V -module is ordinary. According to the definition of rationality, one needs to verify the complete reducibility of any admissible module to prove the rationality. In practice, this is a very difficult task. On the other hand, we need the algebra $A(V)$ to classify the irreducible admissible modules. So the $A(V)$ is available already. While rationality is an external characterization of V , the semisimplicity of $A(V)$ is certainly an internal condition on V as the semisimplicity of an associative algebra can be determined by studying the Jacobson radical. We should point out that the assumption that each irreducible admissible module is ordinary is not strong as this is true for all known simple vertex operator algebras.

The main ideas and tools behind the proof of the main result are the associative

algebras $A_n(V)$ [DLM3] and their bimodules $A_{n,m}(V)$ [DJ1] for nonnegative integers m, n . The associative algebras $A_n(V)$ are generalizations of $A(V)$ such that $A_0(V) = A(V)$. We have already mentioned that $A(V)$ controls the bottom level of an admissible module. The associative algebra $A_n(V)$, for each $n \in \mathbb{Z}_+$, takes care of the first $n + 1$ -levels. Let $M = \bigoplus_{s=0}^{\infty} M(s)$ be an admissible V -module with $M(0) \neq 0$ (see Section 2 for a precise definition). Then each $M(s)$ for $s \leq n$ is a module for $A_n(V)$ [DLM3]. Moreover V is rational if and only if $A_n(V)$ are semisimple for all nonnegative integers n . Theoretically this is a very good result on rationality. But it is very hard to compute $A_n(V)$ for $n > 0$ in practice. Nevertheless, the $A_n(V)$ theory gives a bridge between a vertex operator algebra and associative algebras as far as rationality concerns.

Motivated by the fact that $\text{Hom}_{\mathbb{C}}(M(m), M(n))$ is an $A_n(V)$ - $A_m(V)$ -bimodule, an abstract $A_n(V)$ - $A_m(V)$ -bimodule $A_{n,m}(V)$ is introduced in [DJ1] so that there is a canonical bimodule homomorphism from $A_{n,m}(V)$ to $\text{Hom}_{\mathbb{C}}(M(m), M(n))$ for any admissible V -module M . The most important result about the bimodule theory is an explicit construction of the Verma type admissible V -module generated by any $A_m(V)$ -module U given by

$$M(U) = \bigoplus_{n \geq 0} A_{n,m}(V) \otimes_{A_m(V)} U$$

(see [DJ1] and [DLM3]). As in the classical highest weight module theory of Lie algebras, $M(U)$ has a unique irreducible quotient $L(U)$ in case U is an irreducible $A(V)$ -module. The main idea is to prove that $M(U) = L(U)$ if $A(V)$ is a finite dimensional semisimple associative algebra.

Based on the construction of the Verma type admissible module, a contravariant pairing between $M(U)$ and $M(U^*)$ is constructed for any $A_m(V)$ -module U in this paper. This pairing is an analogue of classical contravariant forms for Kac-Moody Lie algebras and the Virasoro algebra and is used in the proof of the main theorem. In particular, if U is irreducible, then the left radical of the pairing is precisely the maximal proper submodule of $M(U)$. So the contravariant pairing should play an important role in the study of representation theory for vertex operator algebras. It is worthy to mention that this contravariant pairing is totally different from the invariant pairing defined in [FHL] between any admissible module and its graded dual.

As an application of our main result we prove in this paper that the orbifold vertex operator algebra V_L^+ (see [FLM], [AD]) is rational for any positive definite even lattice L . The irreducible admissible modules for V_L^+ have been classified in [DN2] and [AD] and each irreducible admissible module is ordinary. We prove the rationality of V_L^+ by showing that $A(V_L^+)$ is semisimple. In [AD], $A(V_L^+)$ has been understood well enough to classify the irreducible $A(V_L^+)$ -modules. We use a lot of results from [AD] in this paper to prove the semisimplicity of $A(V_L^+)$ and we refer the reader to [AD] for a lot of details. If the rank of L equals one, the rationality of V_L^+ has been obtained [A1] by a different method. Even in this case, it was a very difficult theorem in [A1].

It is hard to avoid the regularity [DLM1] and C_2 -cofiniteness [Z] when dealing with rationality. It has been conjectured that the rationality and regularity are equivalent. There is some progress in proving this conjecture. It is shown in [L] and [ABD] that V is

regular if and only if V is rational and C_2 -cofinite. Since any finitely generated admissible module for a C_2 -cofinite vertex operator algebra is ordinary (see [KL] and [ABD]), an immediate corollary of our result is that V is regular if and only if $A(V)$ is semisimple and C_2 -cofinite. Although only the semisimplicity of $A(V)$ (not the rationality of V) and C_2 -cofiniteness were used to obtain the modular invariance of trace functions [Z] and [DLM4], the rationality of V , in fact, has already been used by our main result.

It is our hope to remove the assumption that each irreducible admissible V -module is ordinary in the near future.

The main results in this paper had been announced in [DJ2].

2 $A_{n,m}(V)$ -theory

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (see [B], [FLM]). A *weak V module* is a pair (M, Y_M) , where M is a vector space and Y_M is a linear map from V to $(\text{End } M)[[z, z^{-1}]]$ satisfying the following axioms for $u, v \in V$, $w \in M$:

$$\begin{aligned} v_n w &= 0 \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large;} \\ Y_M(\mathbf{1}, z) &= \text{id}_M; \end{aligned}$$

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2). \end{aligned}$$

This completes the definition. We denote this module by (M, Y_M) . An *ordinary V -module* is a \mathbb{C} -graded weak V -module

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ is finite and $M_{\lambda+n} = 0$ for fixed λ and $n \in \mathbb{Z}$ small enough, where M_λ is the λ -eigenspace for $L(0)$ with eigenvalue λ and $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

An *admissible V -module* is a weak V -module M which carries a \mathbb{Z}_+ -grading

$$M = \bigoplus_{n \in \mathbb{Z}_+} M(n)$$

(\mathbb{Z}_+ is the set of all nonnegative integers) such that if $r, m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and $a \in V_r$ then $a_m M(n) \subseteq M(r + n - m - 1)$. Since the uniform degree shift gives an isomorphic admissible module we assume $M(0) \neq 0$ in many occasions. It is easy to prove that any ordinary module is an admissible module.

We call a vertex operator algebra *rational* if any admissible module is a direct sum of irreducible admissible modules. It is proved in [DLM2] that if V is rational then there are only finitely many irreducible admissible modules up to isomorphism and each irreducible admissible module is ordinary.

A vertex operator algebra is called *regular* if any weak module is a direct sum of irreducible ordinary modules. It is evident that a regular vertex operator algebra is rational.

We now review the $A_n(V)$ theory from [DLM2]-[DLM3]. Let V be a vertex operator algebra. We define two linear operations $*_n$ and \circ_n on V in the following way:

$$u *_n v = \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_z Y(u, z) \frac{(1+z)^{\text{wt } u+n}}{z^{n+m+1}} v$$

$$u \circ_n v = \text{Res}_z Y(u, z) v \frac{(1+z)^{\text{wt } u+n}}{z^{2n+2}}$$

for homogeneous $u, v \in V$. Let $O_n(V)$ be the linear span of all $u \circ_n v$ and $L(-1)u + L(0)u$. Define the linear space $A_n(V)$ to be the quotient $V/O_n(V)$.

Let M be an admissible V -module. For each homogeneous $v \in V$, we set $o(v) = v_{\text{wt } v-1}$ on M and extend linearly to whole V . The main results on $A_n(V)$ were obtained in [DLM3] (see also [Z] and [DLM2]).

Theorem 2.1. *Let V be a vertex operator algebra and n a nonnegative integer. Then*

- (1) $A_n(V)$ is an associative algebra whose product is induced by $*_n$.
- (2) The identity map on V induces an algebra epimorphism from $A_n(V)$ to $A_{n-1}(V)$.
- (3) Let W be a weak module and set

$$\Omega_n(W) = \{w \in W | u_m w = 0, u \in V, m > \text{wt } u - 1 + n\}.$$

Then $\Omega_n(W)$ is an $A_n(V)$ -module such that $v + O_n(V)$ acts as $o(v)$.

(4) Let $M = \bigoplus_{m=0}^{\infty} M(m)$ be an admissible V -module. Then each $M(m)$ for $m \leq n$ is an $A_n(V)$ -submodule of $\Omega_n(M)$. Furthermore, M is irreducible if and only if each $M(n)$ is an irreducible $A_n(V)$ -module.

(5) For any $A_n(V)$ -module U which is not an $A_{n-1}(V)$ -module, there is a unique Verma type admissible V -module $M(U)$ generated by U so that $M(U)(0) \neq 0$ and $M(U)(n) = U$. Moreover, for any weak V -module W and any $A_n(V)$ -module homomorphism f from U to $\Omega_n(W)$ there is a unique V -homomorphism from $M(U)$ to W which extends f .

(6) V is rational if and only if $A_n(V)$ are finite dimensional semisimple associative algebras for all $n \geq 0$.

(7) If V is rational then there are only finitely many irreducible admissible V -modules up to isomorphism and each irreducible admissible module is ordinary.

(8) The linear map ϕ from V to V defined by $\phi(u) = e^{L(1)}(-1)^{L(0)}u$ for $u \in V$ induces an anti-involution of $A_n(V)$.

The algebra $A(V) = A_0(V)$ has been introduced and studied extensively in [Z] where $*_0$ and \circ_0 were denoted by $*$ and \circ . Theorem 2.1 (1), (3), (5) for $n = 0$ have previously been achieved in [Z] and are very useful in the classification of irreducible modules for vertex operator algebras (see [FZ], [W1], [W2], [KW], [DN1]-[DN3], [Ad], [AD], [DLTY], [TY], [KMY], [A2]). We should also remark that the results in (7) which were parts of

the conditions in the definition of rationality given in [Z] were obtained in [DLM2]. This shows that the rationality defined in this paper and [DLM2] is the same as that defined in [Z].

Next we move to the bimodule theory developed in [DJ1]. For homogeneous $u \in V$, $v \in V$ and $m, n, p \in \mathbb{Z}_+$, define the product $*_{m,p}^n$ on V as follows

$$u *_{m,p}^n v = \sum_{i=0}^p (-1)^i \binom{m+n-p+i}{i} \text{Res}_z \frac{(1+z)^{\text{wt}u+m}}{z^{m+n-p+i+1}} Y(u, z) v.$$

To explain the representation-theoretical meaning of the product $u *_{m,p}^n v$ we consider an admissible V -module $M = \bigoplus_{s \geq 0} M(s)$. For homogeneous $u \in V$ we set $o_t(u) = u_{\text{wt}u-1-t}$ for $t \in \mathbb{Z}$. Then $o_t(u)M(s) \subset M(s+t)$. It is proved in [DJ1] that $o_{n-p}(u)o_{p-m}(v) = o_{n-m}(u *_{m,p}^n v)$ acting on $M(m)$. If $n = p$, we denote $*_{m,p}^n$ by $\bar{*}_m^n$, and if $m = p$, we denote $*_{m,p}^n$ by $*_m^n$, i.e.,

$$u *_{m,p}^n v = \sum_{i=0}^m (-1)^i \binom{n+i}{i} \text{Res}_z \frac{(1+z)^{\text{wt}u+m}}{z^{n+i+1}} Y(u, z) v,$$

$$u \bar{*}_m^n v = \sum_{i=0}^n (-1)^i \binom{m+i}{i} \text{Res}_z \frac{(1+z)^{\text{wt}u+m}}{z^{m+i+1}} Y(u, z) v.$$

Define $O'_{n,m}(V)$ to be the linear span of all $u \circ_m^n v$ and $L(-1)u + (L(0) + m - n)u$, where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_m^n v = \text{Res}_z \frac{(1+z)^{\text{wt}u+m}}{z^{n+m+2}} Y(u, z) v.$$

Then $O_n(V) = O'_{n,n}(V)$ (see [DLM3] and the discussion above). Let $O''_{n,m}(V)$ be the linear span of $u *_{m,p_3}^n ((a *_{p_1,p_2}^{p_3} b) *_{m,p_1}^{p_3} c - a *_{m,p_2}^{p_3} (b *_{m,p_1}^{p_2} c))$, for $a, b, c, u \in V, p_1, p_2, p_3 \in \mathbb{Z}_+$, and $O'''_{n,m}(V) = \sum_{p \in \mathbb{Z}_+} (V *_{p,p}^n O_p(V)) *_{m,p}^n V$. Set

$$O_{n,m}(V) = O'_{n,m}(V) + O''_{n,m}(V) + O'''_{n,m}(V),$$

and

$$A_{n,m}(V) = V/O_{n,m}(V).$$

Here are the main results on $A_{n,m}(V)$ obtained in [DJ1].

Theorem 2.2. *Let V be a vertex operator algebra and m, n nonnegative integers. Then*

(1) *$A_{n,m}(V)$ is an $A_n(V)$ - $A_m(V)$ -bimodule such that the left and right actions of $A_n(V)$ and $A_m(V)$ on $A_{n,m}(V)$ are given by $\bar{*}_m^n$ and $*_m^n$ respectively.*

(2) *The linear map $\phi : V \rightarrow V$ defined by $\phi(u) = e^{L(1)}(-1)^{L(0)}u$ for $u \in V$ induces a linear isomorphism from $A_{n,m}(V)$ to $A_{m,n}(V)$ satisfying the following: $\phi(u \bar{*}_m^n v) = \phi(v) *_{n,m}^m \phi(u)$ and $\phi(v *_{m,n}^n w) = \phi(w) \bar{*}_n^m \phi(v)$ for $u \in A_n(V)$, $w \in A_m(V)$ and $v \in A_{n,m}(V)$.*

(3) Let l be nonnegative integers such that $m - l, n - l \geq 0$. Then $A_{n-l, m-l}(V)$ is an $A_n(V)$ - $A_m(V)$ -bimodule and the identity map on V induces an epimorphism of $A_n(V)$ - $A_m(V)$ -bimodules from $A_{n, m}(V)$ to $A_{n-l, m-l}(V)$.

(4) Define a linear map $\psi: A_{n, p}(V) \otimes_{A_p(V)} A_{p, m}(V) \rightarrow A_{n, m}(V)$ by

$$\psi(u \otimes v) = u *_{m, p}^n v,$$

for $u \otimes v \in A_{n, p}(V) \otimes_{A_p(V)} A_{p, m}(V)$. Then ψ is an $A_n(V) - A_m(V)$ -bimodule homomorphism from $A_{n, p}(V) \otimes_{A_p(V)} A_{p, m}(V)$ to $A_{n, m}(V)$.

(5) Let $M = \bigoplus_{s=0}^{\infty} M(s)$ be an admissible V -module. Then $v + O_{n, m}(V) \mapsto o_{n-m}(v)$ gives an $A_n(V) - A_m(V)$ -bimodule homomorphism from $A_{n, m}(V)$ to $\text{Hom}_{\mathbb{C}}(M(m), M(n))$.

(6) For any $n \geq 0$, the $A_n(V)$ and $A_{n, n}(V)$ are the same.

(7) Let U be an $A_m(V)$ -module which can not factor through $A_{m-1}(V)$. Then

$$\bigoplus_{n \in \mathbb{Z}_+} A_{n, m}(V) \otimes_{A_m(V)} U$$

is a Verma type admissible V -module isomorphic to $M(U)$ given in Theorem 2.1 such that $M(U)(n) = A_{n, m}(V) \otimes_{A_m(V)} U$ with the following universal property: Let W be a weak V -module, then any $A_m(V)$ -morphism $f: U \rightarrow \Omega_m(W)$ can be extended uniquely to a V -homomorphism $f: M(U) \rightarrow W$.

(8) If V is rational and $W^j = \bigoplus_{n \geq 0} W^j(n)$ with $W^j(0) \neq 0$ for $j = 1, 2, \dots, s$ are all the inequivalent irreducible modules of V , then

$$A_{n, m}(V) \cong \bigoplus_{l=0}^{\min\{m, n\}} \left(\bigoplus_{i=1}^s \text{Hom}_{\mathbb{C}}(W^i(m-l), W^i(n-l)) \right).$$

We need the detailed action of V on the Verma type admissible module $M(U)$ generated by an $A_m(V)$ -module U given in Theorem 2.2. For homogeneous $u \in V$ and $p, n \in \mathbb{Z}$, the component operator u_p of $Y_{M(U)}(u, z) = \sum_{p \in \mathbb{Z}} u_p z^{-p-1}$ which maps $M(U)(n)$ to $M(U)(n + \text{wt} u - p - 1)$ is defined by

$$u_p(v \otimes w) = \begin{cases} (u *_{m, n}^{\text{wt} u - p - 1 + n} v) \otimes w, & \text{if } \text{wt} u - 1 - p + n \geq 0, \\ 0, & \text{if } \text{wt} u - 1 - p + n < 0, \end{cases} \quad (2.1)$$

for $v \in A_{n, m}(V)$ and $w \in U$ (see [DJ1] for details).

Theoretically, the construction of the Verma type admissible module $M(U)$ generated by an $A_m(V)$ -module U in [DLM3] is good enough for many purposes. But $M(U)$ constructed in [DLM3] is a quotient module for a certain Lie algebra (see [DLM3]) so it is hard to understand the structure. On the other hand, the construction of $M(U)$ given in Theorem 2.2 (7) is explicit and each homogeneous subspace $M(U)(n)$ is computable. This new construction of $M(U)$ is fundamental in our study of rationality in this paper.

3 Verma type modules

In this section we give a foundational result for this paper. That is, if $A(V)$ is semisimple for a simple vertex operator algebra V then the Verma type admissible V -module generated by any irreducible $A(V)$ -module is irreducible. We need several lemmas.

Recall that if A is an associative algebra and U a left A -module, then the linear dual $U^* = \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ is naturally a right A -module such that $(fa)(u) = f(au)$ for $a \in A$, $f \in U^*$ and $u \in U$.

Lemma 3.1. *Let V be a vertex operator algebra. Assume that $A(V)$ is semisimple and U^i for $i = 1, \dots, s$ are all the inequivalent irreducible $A(V)$ -modules. Let $M(U^i) = \bigoplus_{n \in \mathbb{Z}_+} M(U^i)(n)$ be the Verma type admissible V -module generated by U^i . Then as an $A_n(V)$ - $A(V)$ -bimodule,*

$$A_{n,0}(V) \cong \bigoplus_{i=1}^s M(U^i)(n) \otimes (U^i)^*.$$

Proof: Since $A(V)$ is semisimple, $A_{n,0}(V)$ is a completely reducible right $A(V)$ -module. Note that $\{(U^1)^*, \dots, (U^s)^*\}$ is a complete list of inequivalent irreducible right $A(V)$ -modules. Let $W^i = \text{Hom}_{A(V)}((U^i)^*, A_{n,0}(V))$ which is the multiplicity of $(U^i)^*$ in $A_{n,0}(V)$. Then W^i is a left $A_n(V)$ -module such that $(af)(x) = af(x)$ for $a \in A_n(V)$, $f \in \text{Hom}_{A(V)}((U^i)^*, A_{n,0}(V))$ and $x \in (U^i)^*$ and as $A_n(V)$ - $A(V)$ -bimodules

$$A_{n,0}(V) \cong \bigoplus_{i=1}^s W^i \otimes (U^i)^*.$$

To prove the lemma we need to prove that $W^i = M(U^i)(n)$, $i = 1, \dots, s$. Recall from Theorem 2.2 (7) that for each $1 \leq i \leq s$, $M(U^i)(n)$ is an $A_n(V)$ -module and

$$M(U^i) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,0}(V) \otimes_{A(V)} U^i.$$

Note that $(U^i)^* \otimes_{A(V)} U^j = \delta_{i,j} \mathbb{C}$. We see immediately that W^i and $M(U^i)(n)$ are isomorphic as required. \square

For $m, n, p \in \mathbb{Z}_+$, let

$$A_{n,p}(V) *_{m,p}^n A_{p,m}(V) = \{a *_{m,p}^n b \mid a \in A_{n,p}(V), b \in A_{p,m}(V)\}.$$

Then $A_{n,p}(V) *_{m,p}^n A_{p,m}(V)$ is an $A_n(V) - A_m(V)$ -subbimodule of $A_{n,m}(V)$ by Theorem 2.2 (4). Also recall from Theorem 2.2 (4) the $A_n(V) - A_m(V)$ -bimodule homomorphism $\psi : A_{n,p}(V) \otimes_{A_p(V)} A_{p,m}(V) \rightarrow A_{n,m}(V)$ defined by

$$\psi(u \otimes v) = u *_{m,p}^n v,$$

for $u \in A_{n,p}(V)$ and $v \in A_{p,m}(V)$.

Lemma 3.2. *Let V be a simple vertex operator algebra and $A(V)$ be semisimple. Then for integer $m \geq 2$, we have*

$$A_{0,m}(V) *_{0,m}^0 A_{m,0}(V) = A(V).$$

Proof: It is easy to see that $A_{0,n}(V) *_{0,n}^0 A_{n,0}(V)$ is a two-sided ideal of $A(V)$. Let U be an irreducible module of $A(V)$ and suppose that for some positive integer n ,

$$(A_{0,n}(V) *_{0,n}^0 A_{n,0}(V)) \otimes_{A(V)} U = 0.$$

Let $M(U)$ be the Verma type admissible V -module generated by U and X the admissible V -submodule of $M(U)$ generated by $M(U)(n)$. Then by Proposition 4.5.6 of [LL] (see also [DM1]), X is spanned by $u_p M(u)(n)$ for $u \in V$ and $p \in \mathbb{Z}$. In particular, $X(0)$ is spanned by $u_{wtu-1+n} M(u)(n)$ for $u \in V$. From the module construction of $M(U)$ given in Theorem 2.2 and the action of V on $M(U)$ (2.1) we see that

$$X(0) = (A_{0,n}(V) *_{0,n}^0 A_{n,0}(V)) \otimes_{A(V)} U = 0.$$

So X is a proper admissible submodule of $M(U)$.

Clearly, for any non-zero element u in U , we have $L(1)u = 0$ and $L(-n)u \in M(U)(n)$ from the construction of $M(U)$. Thus $0 = L(1)^n L(-n)u \in X(0)$. It follows that $L(0)u = 0$. This shows that $L(0)U = 0$.

Let $W(U) = M(U)/X$ be the quotient module, then $W(U)(n) = 0$ as $M(U)(n) \subset X$. Since $W(U)(0) = U$, we can assume that $W(U)(n-k) \neq 0$, $W(U)(n-k+1) = 0$, for some positive integer k . Let v be a non-zero element in $W(U)(n-k)$, then $L(-1)v = 0$. By Corollary 4.7.6 and Proposition 4.7.9 of [LL], v is a vacuum-like vector of $W(U)$ and the admissible V -submodule of $W(U)$ generated by v is isomorphic to V . In particular, $L(0)v = 0$. On the other hand, $L(0)v = (n-k)v$ as $L(0)U = 0$. This deduces that $k = n$. Since V_2 contains the Virasoro element $\omega \neq 0$ and $L(-1)$ from V_m to V_{m+1} is injective for any $m \geq 1$, we assert that $V_m \neq 0$ if $m \geq 2$. This implies that $n = 1$. So for any $m \geq 2$, we have

$$(A_{0,m}(V) *_{0,m}^0 A_{m,0}(V)) \otimes U \neq 0.$$

Since $A(V)$ is a finite dimensional semisimple associative algebra, the lemma follows. \square

Lemma 3.3. *Let V be a simple vertex operator algebra such that $A(V)$ is semisimple. Let $n \in \mathbb{Z}_+$ greater than 1, then the $A(V) - A(V)$ -bimodule homomorphism ψ from $A_{0,n}(V) \otimes_{A_n(V)} A_{n,0}(V)$ to $A(V)$ is an isomorphism.*

Proof: By Lemma 3.2, $A_{0,n}(V) *_{0,n}^0 A_{n,0}(V) = A(V)$. So it is enough to prove that ψ is injective. Let $\mathbf{1}$ be the identity of $A(V)$ and $u^j \in A_{0,n}(V)$ and $v^j \in A_{n,0}(V)$, $j = 1, 2, \dots, k$, be such that $\sum_{j=1}^k u^j *_{0,n}^0 v^j = 0$. That is, $\sum_{j=1}^k u^j \otimes_{A_n(V)} v^j$ lies in the kernel of ψ . By Lemma 3.2, there exist $a^i \in A_{0,n}(V)$, $b^i \in A_{n,0}(V)$, $i = 1, 2, \dots, r$ such that

$$\sum_{i=1}^r a^i *_{0,n}^0 b^i = \mathbf{1}.$$

Then

$$\begin{aligned}
\sum_{j=1}^k u^j \otimes v^j &= \left(\sum_{j=1}^k u^j \otimes v^j \right) \cdot \mathbf{1} = \sum_{j=1}^k u^j \otimes (v^j *_{0,0}^n \mathbf{1}) \\
&= \sum_{j=1}^k u^j \otimes (v^j *_{0,0}^n \left(\sum_{i=1}^r a^i *_{0,n}^0 b^i \right)) = \sum_{j=1}^k \sum_{i=1}^r u^j \otimes ((v^j *_{n,0}^n a^i) *_{0,n}^n b^i) \\
&= \sum_{j=1}^k \sum_{i=1}^r (u^j *_{n,n}^0 (v^j *_{n,0}^n a^i)) \otimes b^i = \sum_{j=1}^k \sum_{i=1}^r ((u^j *_{0,n}^0 v^j) *_{n,0}^0 a^i) \otimes b^i \\
&= 0.
\end{aligned}$$

This means that ψ is injective. \square

We are now in a position to prove the following important result on the Verma type admissible module.

Theorem 3.4. *Let V be a simple vertex operator algebra such that $A(V)$ is semisimple. Let U be an irreducible module of $A(V)$, then the Verma type admissible V -module $M(U) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,0}(V) \otimes_{A(V)} U$ generated by U is irreducible.*

Proof: For integer $n \geq 1$, set

$$S(n) = \{x \in A_{n,0}(V) \mid u *_{0,n}^0 x = 0, u \in A_{0,n}(V)\}.$$

Then by the fact that $u *_{0,n}^0 (x *_{0,0}^n v) = (u *_{0,n}^0 x) * v$ and $u *_{0,n}^0 (a *_{0,n}^n x) = (u *_{n,n}^0 a) *_{0,n}^0 x$ for all $u \in A_{0,n}(V)$, $x \in A_{n,0}(V)$, $a \in A_n(V)$, $v \in A(V)$, $S(n)$ is an $A_n(V) - A(V)$ -subbimodule of $A_{n,0}(V)$. The $S(n)$ can be understood naturally from the Verma type admissible V -module

$$M(A(V)) = \bigoplus_{n \geq 0} M(A(V))(n) = \bigoplus_{n \geq 0} A_{n,0}(V) \otimes_{A(V)} A(V) = \bigoplus_{n \geq 0} A_{n,0}(V)$$

generated by $A(V)$. Let S be the maximal submodule of $M(A(V))$ such that

$$S \cap M(A(V))(0) = S \cap A(V) = 0.$$

Then it is clear that $S = \sum_{n \geq 1} S(n)$.

As before, let U^i for $i = 1, \dots, s$ be the inequivalent irreducible modules (which are necessarily finite dimensional) of $A(V)$. Then $A(V) = \bigoplus_{i=1}^s \text{End}_{\mathbb{C}}(U^i)$ and U^i can be considered as a simple left ideal of $A(V)$ and the action of $A(V)$ on U^i is just the multiplication of $A(V)$. Let

$$M(U^i) = \bigoplus_{n \in \mathbb{Z}_+} M(U^i)(n) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,0}(V) \otimes_{A(V)} U^i$$

be the Verma type admissible V -module generated by U^i . Then

$$M(U^i)(n) = A_{n,0}(V) \otimes_{A(V)} U^i = A_{n,0}(V) *_{0,0}^n U^i \subseteq A_{n,0}(V)$$

where we identify U^i with a simple left ideal of $A(V)$. In fact, we can regard $M(U^i)$ as an admissible submodule of $M(A(V))$. The containment $M(U^i)(n) \subset A_{n,0}(V)$ can be understood easily from Lemma 3.1.

Let $J(U^i) = \bigoplus_{n \in \mathbb{Z}_+} J(U^i)(n)$ be the maximal proper admissible V -submodule of $M(U^i)$. Then the quotient module $W^i = M(U^i)/J(U^i) = \bigoplus_{n \in \mathbb{Z}_+} M(U^i)(n)/J(U^i)(n)$ is an irreducible admissible V -module. By Theorem 2.1 (4), $W^i(n)$ is an irreducible $A_n(V)$ -module. Since U^i is an irreducible $A(V)$ -module, we have $J(U^i)(0) = 0$. Regarding $M(U^i)$ as an admissible submodule of $M(A(V))$ then $J(U^i)(n)$ is a subspace of $M(A(V))(n) = A_{n,0}(V) \otimes_{A(V)} A(V) = A_{n,0}(V)$ and

$$A_{0,n}(V) *_{0,n}^0 J(U^i)(n) = 0, \quad (3.1)$$

for all positive integer n . That is, $J(U^i)(n)$ is a subspace of $S(n)$ for all i . To prove that any Verma type admissible V -module $M(U) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,0}(V) \otimes_{A(V)} U$ generated by an irreducible $A(V)$ -module U is irreducible, it is enough to prove that $S(n) = 0$ for all $n \geq 1$. From the assumption, V has only finitely many irreducible admissible modules, so $L(-1)$ from $S(n)$ to $S(n+1)$ is injective if n is large. So it suffices to show that $S(n) = 0$ for all large n .

Claim: If $n \geq 2$, then $(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) = 0$.

If $(A_{n,0}(V) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) = 0$, then $(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) = 0$. So we now assume that $(A_{n,0}(V) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) \neq 0$. By Lemma 3.3 and (3.1) we have for $n \geq 2$ that

$$A_{0,n}(V) \otimes_{A_n(V)} J(U^i)(n) = 0.$$

This implies that

$$A_{0,n}(V) \otimes_{A_n(V)} M(U^i)(n) \cong A_{0,n}(V) \otimes_{A_n(V)} W^i(n). \quad (3.2)$$

Consider the Verma type admissible V -module $M(W^i(n)) = \bigoplus_{m \in \mathbb{Z}_+} A_{m,n}(V) \otimes_{A_n(V)} W^i(n)$ generated by $W^i(n)$. Then $M(W^i(n))(n) = W^i(n)$ is an irreducible $A_n(V)$ -module and $(A_{n,0}(V) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n)$ is a nonzero submodule of $W^i(n)$. This forces $(A_{n,0}(V) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) = W^i(n)$. From the definition of $S(n)$, we have

$$\begin{aligned} & [A_{0,n}(V) *_{n,n}^0 (S(n) *_{n,0}^n A_{0,n}(V))] \otimes_{A_n(V)} W^i(n) \\ &= [(A_{0,n}(V) *_{0,n}^0 S(n)) *_{n,0}^0 A_{0,n}(V)] \otimes_{A_n(V)} W^i(n) = 0. \end{aligned}$$

So $(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n)$ is a proper $A_n(V)$ -submodule of irreducible $A_n(V)$ -module $(A_{n,0}(V) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n)$. Thus

$$(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} W^i(n) = 0.$$

This establishes the claim.

By (3.2), we have

$$(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} M(U^i)(n) = 0, \quad i = 1, 2, \dots, s.$$

Applying Lemma 3.1 yields

$$\begin{aligned}
& (S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} A_{n,0}(V) \\
&= (S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} \left(\bigoplus_{i=1}^s M(U^i)(n) \otimes (U^i)^* \right) \\
&= 0.
\end{aligned}$$

On the other hand, by Lemma 3.2 we have

$$S(n) = S(n) *_{0,0}^n A(V) = S(n) *_{0,0}^n (A_{0,n}(V) *_{0,n}^0 A_{n,0}(V))$$

which is exactly $(S(n) *_{n,0}^n A_{0,n}(V)) \otimes_{A_n(V)} A_{n,0}(V)$ by Lemma 3.3. As a result, $S(n) = 0$. This finishes the proof. \square

We should point out that if $A(V)$ is not semisimple, Theorem 3.4 is false. Here is a counter example. Recall that the abstract Virasoro algebra Vir has a basis $\{L_n | n \in \mathbb{Z}\} \cup \{c\}$ with relation:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c$$

and c is in the center. Then

$$\text{Vir}^{\geq -1} = \bigoplus_{n=-1}^{\infty} \mathbb{C} L_n \oplus \mathbb{C} c$$

is a subalgebra. Given a complex number k , consider the induced module

$$V(k) = U(\text{Vir}) \otimes_{U(\text{Vir}^{\geq -1})} \mathbb{C}_k$$

where $\mathbb{C}_k = \mathbb{C}$ is a module for $\text{Vir}^{\geq -1}$ such that $L_n 1 = 0$ for $n \geq -1$ and $c1 = k$. Then $V(k)$ is a vertex operator algebra (see [FZ]). If $k = 1$, then $V(1)$ is a simple vertex operator algebra as $V(1)$ is an irreducible module for the Virasoro algebra. It is computed in [FZ] that $A(V(1))$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$ which is not semisimple. Let U be an irreducible $A(V(1))$ -module such that ω acts as a positive integer m . Then the Verma type admissible $V(1)$ -module $M(U)$ generated by U is the Verma module $V(1, m) = U(\text{Vir}) \otimes_{U(\text{Vir}^{\geq 0})} \mathbb{C}_{1,m}$ for Vir where $\text{Vir}^{\geq 0} = \sum_{n=0}^{\infty} \mathbb{C} L_n \oplus \mathbb{C} c$ acts on $\mathbb{C}_{1,m} = \mathbb{C}$ in the following way: $L_n 1 = 0$ for $n > 0$ and $L_0 1 = m$, $c1 = 1$. Clearly, $V(1, m)$ is not irreducible (see [KR]).

One can also find counter example from the affine vertex operator algebra if $A(V)$ is not semisimple.

4 Bilinear pairings

Let $M = \bigoplus_{n \geq 0} M(n)$ be an admissible V -module. Then there is a V -invariant bilinear pairing (\cdot, \cdot) between M and its contragradient module $M' = \sum_{n \geq 0} M(n)^*$ in the sense that

$$(Y(u, z)w', w) = (w', Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w)$$

for $u \in V$, $w \in M$ and $w' \in M'$ where $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$ (see [FHL]). The construction of contragradient module M' and the non-degenerate pairing are very useful in the theory of vertex operator algebras. But this bilinear pairing is different from the contravariant form on the Verma modules for affine Lie algebras or the Virasoro algebra when V is an affine or Virasoro vertex operator algebra. While the contravariant form on a Verma module in the classical case is degenerate in general, the invariant bilinear pairing defined in [FHL] is always non-degenerate. In this section we will construct a different invariant bilinear pairing between the two Verma type admissible V -modules $M(U)$ and $M(U^*)$ for any $A_m(V)$ -module U , where $m \in \mathbb{Z}_+$. This pairing is an analogue of classical contravariant forms for Kac-Moody Lie algebras and the Virasoro algebra. This construction heavily depends on the construction of $M(U)$ given in Theorem 2.2 in terms of bimodules.

For $m \in \mathbb{Z}_+$, let U be an $A_m(V)$ -module and recall the anti-involution ϕ of $A_m(V)$ from Theorem 2.1. Then the dual space U^* of U is an $A_m(V)$ -module under the following action:

$$(u \cdot f)(x) = f(\phi(u) \cdot x) = (f, \phi(u) \cdot x), \quad (4.1)$$

for $u \in A_m(V)$, $f \in U^*$ and $x \in U$.

Let $M(U) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$ and $M(U^*) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U^*$ be the Verma type admissible V -modules generated by U and U^* respectively. Recall from Theorem 2.2 the linear isomorphism ϕ from $A_{n,m}(V)$ to $A_{m,n}(V)$. We define a bilinear pairing (\cdot, \cdot) on $M(U^*) \times M(U)$ as follows:

$$(x \otimes f, y \otimes u) = (f, [(\phi(x) *_{m,n}^m y)] \cdot u),$$

for $x, y \in A_{n,m}(V)$, $f \in U^*$, $u \in U$, $n \in \mathbb{Z}_+$; and

$$(A_{p,m}(V) \otimes_{A_m(V)} U^*, A_{n,m}(V) \otimes_{A_m(V)} U) = 0$$

for $p \neq n$. That is $(M(U^*)(p), M(U)(n)) = 0$ if $p \neq n$.

Lemma 4.1. *The bilinear pairing (\cdot, \cdot) is well defined.*

Proof: Let $x \in A_{n,m}(V)$, $y \in A_{p,m}(V)$, $f \in U^*$, $v \in U$, $p, n \in \mathbb{Z}_+$ and $a \in A_m(V)$. If $p = n$, we have from Theorem 2.2 that

$$\begin{aligned} ((x *_{m,m}^n a) \otimes f, y \otimes v) &= (f, [\phi(x *_{m,m}^n a) *_{m,n}^m y] \cdot v) \\ &= (f, [(\phi(a) *_{n,m}^m \phi(x)) *_{m,n}^m y] \cdot v) \\ &= (f, [\phi(a) *_{m,n}^m (\phi(x) *_{m,n}^m y)] \cdot v) \\ &= (f, \phi(a) \cdot [(\phi(x) *_{m,n}^m y) \cdot v]) \\ &= (a \cdot f, (\phi(x) *_{m,n}^m y) \cdot v) \\ &= (x \otimes (a \cdot f), y \otimes v) \end{aligned}$$

where we have used (4.1) in the fifth equality.

If $p \neq n$, it is clear that

$$((x *_{m,m}^n a) \otimes f, y \otimes v) = (x \otimes (a \cdot f), y \otimes v) = 0.$$

Similarly, we have

$$(x \otimes f, (y *_{m,m}^p a) \otimes v) = (x \otimes f, y \otimes (a \cdot v)).$$

The proof is complete. \square

Proposition 4.2. *The bilinear pairing (\cdot, \cdot) on $M(U^*) \times M(U)$ is invariant in the sense that*

$$(Y_M(u, z)(x \otimes f), y \otimes v) = (x \otimes f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})(y \otimes v)) \quad (4.2)$$

for $x \in A_{n,m}(V)$, $y \in A_{p,m}(V)$, $f \in U^*$, $v \in U$, $n, p \in \mathbb{Z}_+$ and $u \in V$.

Proof: It is enough to prove the coefficients of z^{-q-1} in both sides of (4.2) are equal for all $q \in \mathbb{Z}$. That is, we only need to prove

$$(u_q(x \otimes f), y \otimes v) = (x \otimes f, \sum_{j=0}^{\infty} \frac{(-1)^{\text{wt}u}}{j!} (L(1)^j u)_{-q+2\text{wt}u-j-2}(y \otimes v)), \quad (4.3)$$

for $x \in A_{n,m}(V)$, $y \in A_{p,m}(V)$, $f \in U^*$, $v \in U$, $n, p \in \mathbb{Z}_+$, $q \in \mathbb{Z}$ and homogeneous $u \in V$.

First assume that $\text{wt}u - q - 1 + n \neq p$. Since $u_q M(U^*)(n) \subset M(U^*)(\text{wt}u - q - 1 + n)$, then $(u_q M(U^*)(n), M(U)(p)) = 0$ from the definition of (\cdot, \cdot) . Similarly,

$$(M(U^*)(n), \sum_{j=0}^{\infty} \frac{(-1)^{\text{wt}u}}{j!} (L(1)^j u)_{-q+2\text{wt}u-j-2} M(U)(p)) = 0.$$

So (4.3) is true if $\text{wt}u - q - 1 + n \neq p$.

If $\text{wt}u - q - 1 + n = p$, using Theorem 2.2 and the action of u_q given in (2.1) we have

$$\begin{aligned} (u_q(x \otimes f), y \otimes v) &= ((u *_{m,n}^p x) \otimes f, y \otimes v) \\ &= (f, (\phi(u *_{m,n}^p x) *_{m,p}^m y) \cdot v) = (f, ((\phi(x) *_{p,n}^m \phi(u)) *_{m,p}^m y) \cdot v) \\ &= (f, (\phi(x) *_{m,n}^m (\phi(u) *_{m,p}^n y)) \cdot v) = (x \otimes f, (\phi(u) *_{m,p}^n y) \otimes v) \\ &= (x \otimes f, [(\sum_{j=0}^{\infty} \frac{(-1)^{\text{wt}u}}{j!} L(1)^j u) *_{m,p}^n y] \otimes v) \\ &= (x \otimes f, (\sum_{j=0}^{\infty} \frac{(-1)^{\text{wt}u}}{j!} L(1)^j u)_{-q+2\text{wt}u-j-2}(y \otimes v)), \end{aligned}$$

as desired. \square

The proof of the following corollary is standard.

Corollary 4.3. *Let V be a vertex operator algebra and $m \in \mathbb{Z}_+$. Let U be an $A_m(V)$ -module. Then*

$$J(U) = \{w \in M(U) \mid (w', w) = 0, w' \in M(U^*)\}$$

is the maximal proper admissible V -submodule of $M(U)$ such that $J(U) \cap M(U)(m) = 0$. In particular, if U is irreducible then $J(U)$ is the unique maximal proper admissible submodule of $M(U)$.

Corollary 4.3 tells us that the bilinear pairing defined in this section is an analogue of the classical contravariant form in the theory of vertex operator algebras. This result is certainly an important application of the construction of the Verma type admissible V -module $M(U) = \bigoplus_{n \geq 0} A_{n,m}(V) \otimes_{A_m(V)} U$ generated by an $A_m(V)$ -module U . It is hard to imagine how to define the invariant bilinear pairing on $M(U^*) \times M(U)$ without this module construction.

By Theorem 3.4 and Corollary 4.3, we immediately have

Corollary 4.4. *Let V be a simple vertex operator algebra such that $A(V)$ is semisimple. Let U be an irreducible $A(V)$ -module, then the bilinear form (\cdot, \cdot) on $M(U^*) \times M(U)$ is non-degenerate.*

5 Rationality

We prove in this section that a simple vertex operator algebra V is rational if and only if $A(V)$ is semisimple and each irreducible admissible V -module is ordinary. This is the main result of this paper.

Recall from Theorem 2.2 that $A_{m,0}(V) *_{m,0}^m A_{0,m}(V) \subset A_m(V)$. We have

Lemma 5.1. *Let V be a vertex operator algebra. Then*

$$A_{m,0}(V) *_{m,0}^m A_{0,m}(V) \cong O_{m-1}(V)/O_m(V),$$

for $m \in \mathbb{Z}_+ \setminus \{0\}$.

Proof: For $m \in \mathbb{Z}_+ \setminus \{0\}$, let $M(A_m(V)) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,m}(V)$ be the Verma type admissible V -module generated by $A_m(V)$. Then

$$M(A_m(V))(0) = A_{0,m}(V).$$

Let M' be the V -submodule of $M(A_m(V))$ generated by $A_{0,m}(V)$. Then

$$M'(m) = A_{m,0}(V) *_{m,0}^m A_{0,m}(V).$$

Set $W = M(A_m(V))/M'$. Then W is an admissible V -module such that $W(m) = A_m(V)/(A_{m,0}(V) *_{m,0}^m A_{0,m}(V))$ and $W(0) = 0$. So for all homogeneous $u \in V$, we have

$$u_{\text{wt}u-1+m} W(m) = 0.$$

This means that $W(m)$ is an $A_{m-1}(V)$ -module by Theorem 2.1. Thus

$$O_{m-1}(V) *_{m-1} A_m(V) \subset A_{m,0}(V) *_{m,0}^m A_{0,m}(V). \quad (5.1)$$

Note that $A_m(V) = V/O_m(V)$ and $A_{m,0}(V) *_{m,0}^m A_{0,m}(V)$ is the linear span of $u *_{m,0}^m v + O_m(V)$ for all $v \in V$ and homogeneous $u \in V$. Let $B_m(V)$ be the linear span of $u *_{m,0}^m v$ for all $v \in V$ and homogeneous $u \in V$. Then

$$W(m) \cong V/(B_m(V) + O_m(V)), \quad A_{m,0}(V) *_{m,0}^m A_{0,m}(V) = (B_m(V) + O_m(V))/O_m(V).$$

From (5.1), we have

$$O_{m-1}(V) \subseteq B_m(V) + O_m(V).$$

On the other hand, for $v \in V$ and homogeneous $u \in V$

$$u *_{m,0}^m v = \text{Res}_z \frac{(1+z)^{\text{wt}u+m}}{z^{2m+1}} Y(u, z)v \in O_{m-1}(V).$$

That is, $B_m(V) \subset O_{m-1}(V)$. As a result, we have

$$B_m(V) + O_m(V) = O_{m-1}(V),$$

and therefore the lemma holds. \square

Lemma 5.2. *Let V be a vertex operator algebra such that $A(V)$ is semisimple. Let U^i for $i = 1, 2, \dots, s$ be all the inequivalent irreducible modules of $A(V)$. Then for every positive integer n , we have*

$$A_{0,n}(V) \cong \bigoplus_{i=1}^s U^i \otimes M((U^i)^*)(n),$$

where the left action of $A(V)$ on $(U^i)^*$ is defined by (4.1).

Proof: By Lemma 3.1, we have

$$A_{n,0}(V) \cong \bigoplus_{i=1}^s M(U^i)(n) \otimes (U^i)^*.$$

$A_{n,0}(V)$ is an $A_n(V) - A(V)$ -bimodule with the following left and right actions by $A_n(V)$ and $A(V)$ respectively:

$$a \cdot \left(\sum_{i=1}^s u^i \otimes v^i \right) = \sum_{i=1}^s (a \cdot u^i) \otimes v^i, \quad \left(\sum_{i=1}^s u^i \otimes v^i \right) \cdot b = \sum_{i=1}^s u^i \otimes v^i \cdot b,$$

where $a \in A_n(V)$, $b \in A(V)$, $u^i \in M(U^i)(n)$, $v^i \in (U^i)^*$, and $(v^i \cdot b)(c) = v^i(b \cdot c)$, for $c \in U^i$. By Theorem 3.4, each $M(U^i)(n) \otimes (U^i)^*$ is an irreducible $A_n(V) - A(V)$ -bimodule.

Let $m, n \in \mathbb{Z}_+$, recall from Theorem 2.2 that ϕ is the linear map from $A_{n,m}(V)$ to $A_{m,n}(V)$ defined as $\phi(u) = e^{L(1)}(-1)^{L(0)}u$, for $u \in A_{n,m}(V)$. By Proposition 3.2 of [DJ1], ϕ ,

in fact, is an isomorphism of $A_n(V) - A_m(V)$ -bimodules from $A_{n,m}(V)$ to $A_{m,n}(V)$, where the left action \cdot_m^n of $A_n(V)$ and the right action \cdot_m^n of $A_m(V)$ on $A_{m,n}(V)$ are defined by $u \cdot_m^n v = v *_{\bar{n}}^m \phi(u)$ and $v \cdot_m^n a = \phi(a) *_{\bar{n}}^m v$ respectively for $u \in A_n(V)$, $v \in A_{m,n}(V)$ and $a \in A_m(V)$.

So it is sufficient to show that $A_{n,0}(V)$ is isomorphic to $\bigoplus_{i=1}^s U^i \otimes M((U^i)^*)(n)$ as $A(V) - A_n(V)$ -bimodule with the new actions. As an $A(V) - A_n(V)$ -bimodule $M(U^i)(n) \otimes (U^i)^*$ is clearly isomorphic to $(U^i)^* \otimes M(U^i)(n)$. Since each U^i is finite dimensional we finish the proof. \square

We now prove the main theorem.

Theorem 5.3. *Let V be a simple vertex operator algebra. Then V is rational if and only if $A(V)$ is semisimple and each irreducible admissible V -module is ordinary.*

Proof: By Theorem 2.1, if V is rational then $A(V)$ is semisimple and each irreducible admissible V -module is ordinary. Now we assume that $A(V)$ is semisimple and each irreducible admissible V -module is ordinary. By Theorem 2.1, it is good enough to prove that $A_n(V)$ are semisimple for all positive integers n . We achieve this by induction on n .

Suppose $n \geq 1$ such that $A_m(V)$ are all semisimple for $0 \leq m \leq n-1$. In order to prove that $A_n(V)$ is semisimple, it is good enough to prove that any indecomposable $A_n(V)$ -module X is irreducible. We may assume that X is not an $A_{n-1}(V)$ -module.

By Lemma 3.1 and Lemma 5.2, both $A_{n,0}(V)$ and $A_{0,n}(V)$ are finite dimensional (here we are using the assumption that each irreducible admissible module is ordinary). It follows from Lemma 5.1 that $O_{n-1}(V)/O_n(V)$ is finite dimensional. Since $A_{n-1}(V)$ is also finite dimensional by inductive assumption, we see that $A_n(V)$ is finite dimensional. In fact, the dimension of $A_n(V)$ is the sum of dimensions of $O_{n-1}(V)/O_n(V)$ and $A_{n-1}(V)$. This implies that X is finite dimensional.

Consider the Verma type admissible V -module $M(X) = \bigoplus_{k \geq 0} A_{k,n}(V) \otimes_{A_n(V)} X$. Then by Theorem 2.1, $M(X)(0) = A_{0,n}(V) \otimes_{A_n(V)} X \neq 0$. By assumption, $M(X)(0)$ is a semisimple $A(V)$ -module. Let Z be the submodule of $M(X)$ generated by $M(X)(0)$. Then Z is completely reducible by Theorem 3.4 and $Z(n) = X \cap Z$ is a completely reducible $A_n(V)$ -submodule of X . Then $X/Z(n)$ is an $A_{n-1}(V)$ -module.

By the inductive assumption $X/Z(n)$ is a direct sum of irreducible $A_{n-1}(V)$ -modules. Without loss, we can assume that $X/Z(n)$ is an irreducible $A_{n-1}(V)$ -module. Then $X/Z(n)$ must be finite dimensional as $A_{n-1}(V)$ is semisimple.

On the other hand, X^* is also an $A_n(V)$ -module and $(X/Z(n))^*$ is a submodule. Let $U = M(X)(0)^*$. Then the Verma type admissible module $M(U) = \bigoplus_{m \geq 0} A_{m,0}(V) \otimes_{A(V)} U$ is completely reducible and $M(U)(n)$ is a submodule of X^* . As a result we see that $X^* = M(U)(n) \oplus (X/Z(n))^*$. Since X is indecomposable, X^* is also indecomposable. This shows that $X/Z(n) = 0$ and $X = Z(n)$ is irreducible.

Here we give *another proof* for the statement that if $A(V)$ is semisimple and each irreducible admissible V -module is ordinary then V is rational. Again we will prove that $A_n(V)$ are semisimple for all n . We assume that $A_m(V)$ are semisimple for all $0 \leq m < n$.

First by Lemma 3.1, Lemma 5.2 and Corollary 4.4 we have

$$\begin{aligned}
A_{n,0}(V) \otimes_{A(V)} A_{0,n}(V) &= \bigoplus_{i,j=1}^s M(U^i)(n) \otimes (U^i)^* \otimes_{A(V)} U^j \otimes M((U^j)^*)(n) \\
&= \bigoplus_{i=1}^s M(U^i)(n) \otimes M((U^i)^*)(n) \\
&= \bigoplus_{i=1}^s \text{End}(M(U^i)(n)).
\end{aligned}$$

In particular, the dimension of $A_{n,0}(V) \otimes_{A(V)} A_{0,n}(V)$ is $\sum_{i=1}^s (\dim M(U^i)(n))^2$.

By Theorem 2.2, $A_{n,0}(V) *_{n,0}^n A_{0,n}(V)$ is a homomorphic image of $A_{n,0}(V) \otimes_{A(V)} A_{0,n}(V)$. So its dimension is less than or equal to $\sum_{i=1}^s (\dim M(U^i)(n))^2$.

From Lemma 5.1 we see that

$$\begin{aligned}
\dim A_n(V) &= \dim A_{n-1}(V) + \dim(O_n(V)/O_{n-1}(V)) \\
&= \dim A_{n-1}(V) + \dim A_{n,0}(V) *_{n,0}^n A_{0,n}(V) \\
&\leq \sum_{i=1}^s \sum_{m=0}^n (\dim M(U^i)(m))^2
\end{aligned}$$

where we have used the fact that

$$\dim A_{n-1}(V) = \sum_{i=1}^s \sum_{m=0}^{n-1} (\dim M(U^i)(m))^2$$

as $A_{n-1}(V)$ is semisimple.

On the other hand, $\{M(U^i)(m) | i = 1, \dots, s, m = 0, \dots, n\}$ are the inequivalent irreducible modules for finite dimensional algebra $A_n(V)$ by Theorem 2.1. So the dimension of $A_n(V)$ is at least $\sum_{i=1}^s \sum_{m=0}^n (\dim M(U^i)(m))^2$. This forces

$$\dim A_n(V) = \sum_{i=1}^s \sum_{m=0}^n (\dim M(U^i)(m))^2.$$

Thus

$$A_n(V) \cong \bigoplus_{i=1}^s \bigoplus_{m=0}^n \text{End}(M(U^i)(m))$$

is semisimple. \square

We remark that if the Verma type module $M(U)$ generated by any irreducible $A(V)$ -module U is irreducible, $A(V)$ is semisimple and V is C_2 -cofinite it is proved in [DLTY] that V is rational. It is clear that the assumptions in Theorem 5.3 are much weaker as any irreducible admissible module is ordinary for any C_2 -cofinite vertex operator algebra (cf. [KL], [Bu], [ABD]).

This theorem has several corollaries. Following [KL] we call V C_1 -cofinite if $V = \sum_{n \geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$ satisfies that $C_1(V)$ has finite codimension in V , where $C_1(V)$ is spanned by $u_{-1}v$ and $L(-1)u$ for all $u, v \in \sum_{n > 0} V_n$. It is proved in [KL] that if V is C_1 -cofinite then any irreducible admissible V -module is ordinary.

Corollary 5.4. *If V is a C_1 -cofinite simple vertex operator algebra such that $A(V)$ is semisimple then V is rational.*

A vertex operator algebra V is called C_2 -cofinite if the subspace $C_2(V)$ spanned by $u_{-2}v$ for all $u, v \in V$ has finite codimension in V [Z]. The C_2 -cofiniteness has played a very important role in the theory of vertex operator algebras and conformal field theory (see [Z], [DLM4], [GN], [NT], [DM2], [H], [DM3]). It is proved in [L] and [ABD] that V is regular if and only if V is rational and C_2 -cofinite. This implies the following corollary.

Corollary 5.5. *Let V be a simple C_2 -cofinite vertex operator algebra. Then the following are equivalent:*

- (a) V is rational.
- (b) V is regular.
- (c) $A(V)$ is semisimple.

6 An application: rationality of V_L^+

We prove in this section the rationality of V_L^+ for all positive definite even lattice L . If the rank of L is 1, the rationality has been established previously in [A1].

Let L be a positive definite even lattice of rank d and V_L the vertex operator algebra associated to L (cf. [B], [FLM]). Let V_L^+ be the fixed points of V_L under the automorphism θ lifted from the -1 isometry of L . Then V_L^+ is a simple vertex operator subalgebra of V_L . If $d = 1$ it is proved in [A1] that V_L^+ is rational. In this section we extend the rationality to all V_L^+ . In [DN2] and [AD] we classify the irreducible admissible modules of V_L^+ . It turns out that all the irreducible admissible modules are ordinary. Based on the work [DN2] and [AD], we prove in this section that $A(V_L^+)$ is semisimple. Thus by Theorem 5.3, V_L^+ is rational.

We use the setting of [FLM]. In particular, \widehat{L} is the canonical central extension of L by the cyclic group $\langle \kappa \rangle$ of order 2: $1 \rightarrow \langle \kappa \rangle \rightarrow \widehat{L} \rightarrow L \rightarrow 0$ with the commutator map $c(\alpha, \beta) = \kappa^{(\alpha, \beta)}$ for $\alpha, \beta \in L$. Let $e : L \rightarrow \widehat{L}$ be a section such that $e_0 = 1$ and $\epsilon : L \times L \rightarrow \langle \kappa \rangle$ the corresponding 2-cocycle. We can assume that ϵ is bimultiplicative. Then $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = \kappa^{(\alpha, \beta)}$,

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma),$$

and $e_\alpha e_\beta = \epsilon(\alpha, \beta)e_{\alpha+\beta}$ for $\alpha, \beta, \gamma \in L$. Let θ denote the automorphism of \widehat{L} defined by $\theta(e_\alpha) = e_{-\alpha}$ and $\theta(\kappa) = \kappa$. Set $K = \{a^{-1}\theta(a) | a \in \widehat{L}\}$.

Let $M(1)$ be the Heisenberg vertex operator algebra associated to $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. Then $V_L = M(1) \otimes \mathbb{C}[L]$ where $\mathbb{C}[L]$ is the group algebra of L with a basis e^α for $\alpha \in L$ and is an \widehat{L} -module such that $e_\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}$.

Recall that $L^\circ = \{\lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z}\}$ is the dual lattice of L . There is an \widehat{L} -module structure on $\mathbb{C}[L^\circ] = \bigoplus_{\lambda \in L^\circ} \mathbb{C}e^\lambda$ such that κ acts as -1 (see [DL]). Let $L^\circ = \bigcup_{i \in L^\circ/L} (L + \lambda_i)$ be the coset decomposition such that $\lambda_0 = 0$. Set $\mathbb{C}[L + \lambda_i] = \bigoplus_{\alpha \in L} \mathbb{C}e^{\alpha + \lambda_i}$. Then $\mathbb{C}[L^\circ] = \bigoplus_{i \in L^\circ/L} \mathbb{C}[L + \lambda_i]$ and each $\mathbb{C}[L + \lambda_i]$ is an \widehat{L} -submodule of $\mathbb{C}[L^\circ]$. The action of \widehat{L} on $\mathbb{C}[L + \lambda_i]$ is as follows:

$$e_\alpha e^{\beta + \lambda_i} = \epsilon(\alpha, \beta) e^{\alpha + \beta + \lambda_i}$$

for $\alpha, \beta \in L$. On the surface, the module structure on each $\mathbb{C}[L + \lambda_i]$ depends on the choice of λ_i in $L + \lambda_i$. It is easy to prove that different choices of λ_i give isomorphic \widehat{L} -modules.

Set $\mathbb{C}[M] = \bigoplus_{\lambda \in M} \mathbb{C}e^\lambda$ for a subset M of L° , and define $V_M = M(1) \otimes \mathbb{C}[M]$. Then V_L is a rational vertex operator algebra and $V_{L + \lambda_i}$ for $i \in L^\circ/L$ are the irreducible modules for V_L (see [B], [FLM], [D1], [DLM1]).

Define a linear isomorphism $\theta : V_{L + \lambda_i} \rightarrow V_{L - \lambda_i}$ for $i \in L^\circ/L$ by

$$\theta(\beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)e^{\alpha + \lambda_i}) = (-1)^k \beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)e^{-\alpha - \lambda_i}$$

for $\beta_i \in \mathfrak{h}$, $n_i \geq 1$ and $\alpha \in L$ if $2\lambda_i \notin L$, and

$$\begin{aligned} \theta(\beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)e^{\alpha + \lambda_i}) \\ = (-1)^k c_{2l_i} \epsilon(\alpha, 2\lambda_i) \beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)e^{-\alpha - \lambda_i} \end{aligned}$$

if $2\lambda_i \in L$ where c_{2l_i} is a square root of $\epsilon(2l_i, 2l_i)$. Then θ defines a linear isomorphism from V_{L° to itself such that

$$\theta Y(u, z)v = Y(\theta u, z)\theta v$$

for $u \in V_L$ and $v \in V_{L^\circ}$. In particular, θ is an automorphism of V_L which induces an automorphism of $M(1)$.

For any θ -stable subspace U of V_{L° , let U^\pm be the ± 1 -eigenspace of U for θ . Then V_L^\pm is a simple vertex operator algebra.

Also recall the θ -twisted Heisenberg algebra $\mathfrak{h}[-1]$ and its irreducible module $M(1)(\theta)$ from [FLM] and [AD]. Let χ be a central character of \widehat{L}/K such that $\chi(\kappa) = -1$ and T_χ the irreducible \widehat{L}/K -module with central character χ . Then $V_L^{T_\chi}$ is an irreducible θ -twisted V_L -module (see [FLM], [D2] and [DL]). We also define an action of θ on $V_L^{T_\chi}$ such that

$$\theta(\beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)t) = (-1)^k \beta_1(-n_1)\beta_2(-n_2) \cdots \beta_k(-n_k)t$$

for $\beta_i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_+$ and $t \in T_\chi$. Recall that $L^\circ = \bigcup_{i \in L^\circ/L} (L + \lambda_i)$. Here is the classification of irreducible modules for V_L^\pm (see [DN2] and [AD]).

Theorem 6.1. *Let L be a positive definite even lattice. Then any irreducible admissible V_L^\pm -module is isomorphic to one of irreducible modules V_L^\pm , $V_{L + \lambda_i}(2\lambda_i \notin L)$, $V_{L + \lambda_i}^\pm(2\lambda_i \in L)$ and $V_L^{T_\chi, \pm}$ for any irreducible \widehat{L}/K -module T_χ with central character χ .*

Then the irreducible $A(V_L^+)$ -modules are the top levels $W(0)$ of irreducible V_L^+ -modules W given as follows:

$$\begin{aligned} V_L^+(0) &= \mathbb{C}\mathbf{1}, \quad V_L^-(0) = \mathfrak{h}(-1) \bigoplus \left(\bigoplus_{\alpha \in L_2} \mathbb{C}(e^\alpha - e^{-\alpha}) \right), \\ V_{\lambda_i+L}(0) &= \bigoplus_{\alpha \in \Delta(\lambda_i)} \mathbb{C}e^{\lambda_i+\alpha} \quad (2\lambda_i \notin L), \\ V_{\lambda_i+L}^\pm(0) &= \sum_{\alpha \in \Delta(\lambda_i)} \mathbb{C}(e^{\lambda_i+\alpha} \pm \theta e^{\lambda_i+\alpha}) \quad (2\lambda_i \in L), \\ V_L^{T_\chi,+}(0) &= T_\chi, \quad V_L^{T_\chi,-}(0) = \mathfrak{h}(-1/2) \otimes T_\chi. \end{aligned}$$

Here $\mathfrak{h}(-1) = \{h(-1) | h \in \mathfrak{h}\} \subset M(1)$ and $\mathfrak{h}(-1/2) = \{h(-1/2) | h \in \mathfrak{h}\} \subset M(1)(\theta)$.

Let $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} . Recall from [DN3] and [AD] the following vectors in V_L^+ for $a, b = 1, \dots, d$ and $\alpha \in L$

$$\begin{aligned} S_{ab}(m, n) &= h_a(-m)h_b(-n), \\ E_{ab}^u &= 5S_{ab}(1, 2) + 25S_{ab}(1, 3) + 36S_{ab}(1, 4) + 16S_{ab}(1, 5) \quad (a \neq b), \\ \bar{E}_{ba}^u &= S_{ab}(1, 1) + 14S_{ab}(1, 2) + 41S_{ab}(1, 3) + 44S_{ab}(1, 4) + 16S_{ab}(1, 5) \quad (a \neq b), \\ E_{aa}^u &= E_{ab}^u E_{ba}^u, \\ E_{ab}^t &= -16(3S_{ab}(1, 2) + 14S_{ab}(1, 3) + 19S_{ab}(1, 4) + 8S_{ab}(1, 5)) \quad (a \neq b), \\ \bar{E}_{ba}^t &= -16(5S_{ab}(1, 2) + 18S_{ab}(1, 3) + 21S_{ab}(1, 4) + 8S_{ab}(1, 5)) \quad (a \neq b), \\ E_{aa}^t &= E_{ab}^t E_{ba}^t, \\ \Lambda_{ab} &= 45S_{ab}(1, 2) + 190S_{ab}(1, 3) + 240S_{ab}(1, 4) + 96S_{ab}(1, 5), \\ E^\alpha &= e^\alpha + e^{-\alpha}. \end{aligned}$$

For $v \in V_L^+$ we denote $v + O(V_L^+)$ by $[v]$. Let A^u and A^t be the linear subspace of $A(M(1)^+)$ spanned by $E_{ab}^u + O(M(1)^+)$ and $E_{ab}^t + O(M(1)^+)$ respectively for $1 \leq a, b \leq d$. Then A^t and A^u are two sided ideals of $A(M(1)^+)$. Note that the natural algebra homomorphism from $A(M(1)^+)$ to $A(V_L^+)$ gives embedding of A^u and A^t into $A(V_L^+)$. We should remark that the A^u and A^t are independent of the choice of the orthonormal basis $\{h_1, \dots, h_d\}$.

By Lemma 7.3 of [AD] we know that

$$V_L^-(0) = \mathfrak{h}(-1) \bigoplus \left(\sum_{\alpha \in L_2} \mathbb{C}[E^\alpha] \alpha(-1) \right),$$

where $L_2 = \{\alpha \in L | (\alpha, \alpha) = 2\}$. Let $L_2 = \{\pm\alpha_1, \dots, \pm\alpha_r, \pm\alpha_{r+1}, \dots, \pm\alpha_{r+l}\}$ be such that $\{\alpha_1, \dots, \alpha_r\}$ are linearly independent and $\{\alpha_{r+1}, \dots, \alpha_{r+l}\} \subseteq \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$. We can choose the orthonormal basis $\{h_i | i = 1, \dots, d\}$ so that $h_i \in \mathbb{C}\alpha_1 + \dots + \mathbb{C}\alpha_i$, for $i = 1, \dots, r$. Then we have

$$\begin{aligned} \alpha_i(-1) &= a_{i1}h_1(-1) + \dots + a_{ii}h_i(-1), \quad i = 1, \dots, r, \\ \alpha_j(-1) &= a_{j1}h_1(-1) + \dots + a_{jr}h_r(-1), \quad j = r+1, \dots, r+l, \end{aligned}$$

where $a_{ii} \neq 0, i = 1, \dots, r$. For $i \in \{1, 2, \dots, l\}$, let k_i be such that

$$a_{r+i, k_i} \neq 0, \quad a_{r+i, k_i+1} = \dots = a_{r+i, r} = 0.$$

We know from [AD] that $e^i = h_i(-1)$ for $i = 1, \dots, d$ and $e^{d+j} = [E^{\alpha_j}] \alpha_j(-1)$ for $j = 1, \dots, r+l$ form a basis of $V_L^-(0)$. We first construct a two-sided ideal of $A(V_L^+)$ isomorphic to $\text{End}(V_L^-(0))$. Recall E_{ij}^u for $i, j = 1, \dots, d$. We now extend the definition of E_{ij}^u to all $i, j = 1, \dots, d+r+l$ and the linear span of E_{ij}^u will be the ideal of $A(V_L^+)$ isomorphic to $\text{End} V_L^-(0)$ (with respect to the basis $\{e^1, \dots, e^{d+r+l}\}$).

For the notational convenience, we also write $E_{i,j}^u$ for E_{ij}^u from now on. Define

$$[E_{j,d+i}^u] = \frac{1}{4\epsilon(\alpha_i, \alpha_i)a_{ii}}[E_{ji}^u * E^{\alpha_i}], \quad i = 1, \dots, r, j = 1, \dots, d,$$

$$[E_{j,d+r+i}^u] = \frac{1}{4\epsilon(\alpha_{r+i}, \alpha_{r+i})a_{r+i, k_i}}[E_{j, k_i}^u * E^{\alpha_{r+i}}], \quad i = 1, \dots, l, j = 1, \dots, d.$$

Define

$$[E_{d+i, j}^u] = \sum_{k=1}^r a_{ik}[E^{\alpha_i}] * [E_{kj}^u], \quad i = 1, \dots, r+l, j = 1, \dots, d,$$

where $a_{ij} = 0$, for $1 \leq i < j \leq r$. Recall from [DN3] and [AD] that $[E_{ab}^u]h_c(-1) = \delta_{c,b}h_a(-1)$ for $a, b, c = 1, \dots, d$.

Lemma 6.2. *The following holds:*

$$[E_{ij}^u]e^k = \delta_{k,j}e^i, \quad [E_{st}^u]e^k = \delta_{t,k}e^s$$

for $i, t = 1, \dots, d, j, s = d+1, \dots, d+r+l$ and $k = 1, \dots, d+r+l$.

Proof: Let $h \in \mathfrak{h}$ such that $(h, h) \neq 0$. Then $\omega_h = \frac{1}{2(h, h)}h(-1)^2$ is a Virasoro element with central charge 1. Note that $\omega_h\beta(-1) = \frac{(\beta, h)^4}{2(h, h)}h(-1)$ for any $\beta \in \mathfrak{h}$. For $\alpha \in L_2$ then $[E^\alpha] * [E^\alpha] = 4\epsilon(\alpha, \alpha)[\omega_\alpha]$ in $A(V_L^+)$ by Proposition 4.9 of [AD]. Then for $i = 1, \dots, d, j = 1, \dots, r$, we have

$$\begin{aligned} [E_{i,d+j}^u]e^{d+j} &= [E_{i,d+j}^u]([E^{\alpha_j}]\alpha_j(-1)) \\ &= \frac{1}{4\epsilon(\alpha_j, \alpha_j)a_{jj}}([E_{ij}^u] * [E^{\alpha_j}] * [E^{\alpha_j}])\alpha_j(-1) \\ &= \frac{1}{a_{jj}}[E_{ij}^u]\alpha_j(-1) = h_i(-1). \end{aligned}$$

Let $k \in \{1, \dots, r+l\}$ such that $k \neq j$. Then

$$\begin{aligned} [E_{i,d+j}^u]e^{d+k} &= [E_{i,d+j}^u]([E^{\alpha_k}]\alpha_k(-1)) \\ &= \frac{1}{4\epsilon(\alpha_j, \alpha_j)a_{jj}}([E_{ij}^u] * [E^{\alpha_j}] * [E^{\alpha_k}])\alpha_k(-1). \end{aligned}$$

By Proposition 5.4 of [AD], we have

$$[E^{\alpha_j}] * [E^{\alpha_k}] = \sum_p [v^p] * [E^{\alpha_j + \alpha_k}] * [w^p] + \sum_q [x^q] * [E^{\alpha_j - \alpha_k}] * [y^q],$$

where $v^p, w^p, x^q, y^q \in M(1)^+$. Since A^u is an ideal of $A(M(1)^+)$, we have $[E_{ji}^u] * [v^p], [E_{ji}^u] * [x^q] \in A^u$. By the proof of Proposition 7.2 of [AD], we know that $A^u[E^\alpha]\alpha(-1) = 0$, for any $\alpha \in L_2$. So by Lemma 7.1 and Proposition 7.2 of [AD], we have

$$[E_{i,d+j}^u]e^{d+k} = 0, \quad i = 1, \dots, d, j = 1, \dots, r, k = 1, \dots, r+l, j \neq k.$$

It follows from the proof of Proposition 7.2 of [AD] that

$$E_{i,j+d}^u e^s = [E_{ij}^u] * [E^{\alpha_j}] h_s(-1) = 0$$

for $s = 1, \dots, d$ as $[E^{\alpha_j}] h_s(-1) \in \sum_{p=1}^{r+l} \mathbb{C}[e^{\alpha_p} - e^{-\alpha_p}]$. This completes the proof for $E_{i,j+d}^u$ for $i = 1, \dots, d, j = 1, \dots, r$. The other cases can be done similarly. \square

Recall H_a and $\omega_a = \omega_{h_a}$ for $a = 1, \dots, d$ from [DN3] and [AD]. The following lemma collects some formulas from Propositions 4.5, 4.6, 4.8 and 4.9 of [AD].

Lemma 6.3. *For any indices a, b, c, d ,*

$$[\omega_a] * [E_{bc}^u] = \delta_{ab} [E_{bc}^u], \quad (6.1)$$

$$[E_{bc}^u] * [\omega_a] = \delta_{ac} [E_{bc}^u]. \quad (6.2)$$

$$[E_{ab}^u] * [E_{cd}^t] = [E_{cd}^t] * [E_{ab}^u] = 0, \quad (6.3)$$

$$[\Lambda_{ab}] * [E_{cd}^u] = [\Lambda_{ab}] * [E_{cd}^t] = [E_{cd}^u] * [\Lambda_{ab}] = [E_{cd}^t] * [\Lambda_{ab}] = 0 \quad (a \neq b), \quad (6.4)$$

For distinct a, b, c ,

$$(70[H_a] + 1188[\omega_a]^2 - 585[\omega_a] + 27) * [H_a] = 0, \quad (6.5)$$

$$([\omega_a] - 1) * \left([\omega_a] - \frac{1}{16}\right) * \left([\omega_a] - \frac{9}{16}\right) * [H_a] = 0, \quad (6.6)$$

$$-\frac{2}{9}[H_a] + \frac{2}{9}[H_b] = 2[E_{aa}^u] - 2[E_{bb}^u] + \frac{1}{4}[E_{aa}^t] - \frac{1}{4}[E_{bb}^t], \quad (6.7)$$

$$\begin{aligned} & -\frac{4}{135}(2[\omega_a] + 13) * [H_a] + \frac{4}{135}(2[\omega_b] + 13) * [H_b] \\ & = 4([E_{aa}^u] - [E_{bb}^u]) + \frac{15}{32}([E_{aa}^t] - [E_{bb}^t]), \end{aligned} \quad (6.8)$$

$$[\omega_b] * [H_a] = -\frac{2}{15}([\omega_a] - 1) * [H_a] + \frac{1}{15}([\omega_b] - 1) * [H_b], \quad (6.9)$$

$$[\Lambda_{ab}]^2 = 4[\omega_a] * [\omega_b] - \frac{1}{9}([H_a] + [H_b]) - ([E_{aa}^u] + [E_{bb}^u]) - \frac{1}{4}([E_{aa}^t] + [E_{bb}^t]), \quad (6.10)$$

$$[\Lambda_{ab}] * [\Lambda_{bc}] = 2[\omega_b] * [\Lambda_{ac}]. \quad (6.11)$$

For $\alpha \in L$ such that $(\alpha, \alpha) = 2k \neq 2$,

$$[H_\alpha] * [E^\alpha] = \frac{18(8k-3)}{(4k-1)(4k-9)} \left([\omega_\alpha] - \frac{k}{4} \right) \left([\omega_\alpha] - \frac{3(k-1)}{4(8k-3)} \right) [E^\alpha], \quad (6.12)$$

$$\left([\omega_\alpha] - \frac{k}{4} \right) \left([\omega_\alpha] - \frac{1}{16} \right) \left([\omega_\alpha] - \frac{9}{16} \right) [E^\alpha] = 0. \quad (6.13)$$

If $\alpha \in L_2$,

$$[E^\alpha] * [E^\alpha] = 4\epsilon(\alpha, \alpha)[\omega_\alpha], \quad (6.14)$$

$$[H_\alpha] * [E^\alpha] + [E^\alpha] * [H_\alpha] = -12[\omega_\alpha] * \left([\omega_\alpha] - \frac{1}{4} \right) * [E^\alpha], \quad (6.15)$$

$$([\omega_\alpha] - 1) * \left([\omega_\alpha] - \frac{1}{4} \right) * \left([\omega_\alpha] - \frac{1}{16} \right) * \left([\omega_\alpha] - \frac{9}{16} \right) * [E^\alpha] = 0. \quad (6.16)$$

For any $\alpha \in L$,

$$I^t * [E^\alpha] = [E^\alpha] * I^t, \quad (6.17)$$

where I^t is the identity of the simple algebra A^t .

Lemma 6.4. For any $\alpha \in L_2$, we have

$$A^u * [E^\alpha] * A^u = 0.$$

Proof: Let $\alpha \in L_2$ and $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} such that $h_1 \in \mathbb{C}\alpha$. (A^u is independent of the choice of orthonormal basis.) By (6.1)-(6.2) and (6.16), we have $[E^\alpha] = f([\omega_\alpha]) * [E^\alpha] = [E^\alpha] * f([\omega_\alpha])$ for some polynomial $f(x)$ with $f(0) = 0$. Note that $\omega_\alpha = \omega_1$. By (6.1)-(6.2), we only need to prove that

$$[E_{i1}^u] * [E^\alpha] * [E_{1s}^u] = 0, \quad i, s = 1, 2, \dots, d.$$

Let $a = 1, b \neq 1$ in (6.9). Multiplying (6.9) by $[E_{i1}^u]$ on left and using (6.2) and (6.3), we have

$$[E_{i1}^u] * [H_b] = 0, \quad b \neq 1.$$

Then setting $a = 1, b \neq 1$ in (6.7) and multiplying (6.7) by $[E_{i1}^u]$ on left yields

$$[E_{i1}^u] * [H_1] = -9[E_{i1}^u].$$

Let $a = 1, b \neq 1$ in (6.10). Multiplying (6.10) by $[E_{1s}^u]$ on right and using (6.1) and (6.4), we have

$$-\frac{1}{9}[H_1] * [E_{1s}^u] - \frac{1}{9}[H_b] * [E_{1s}^u] = [E_{1s}^u].$$

On the other hand, multiplying (6.7) by $[E_{1s}^u]$ on right yields

$$-\frac{1}{9}[H_1] * [E_{1s}^u] + \frac{1}{9}[H_b] * [E_{1s}^u] = [E_{1s}^u].$$

Comparing the above two formulas, we have

$$[H_1] * [E_{1s}^u] = -9[E_{1s}^u], [H_a] * [E_{1s}^u] = 0, a \neq 1.$$

So

$$[E_{i1}^u] * [H_1] * [E^\alpha] * [E_{1s}^u] + [E_{i1}^u] * [E^\alpha] * [H_1] * [E_{1s}^u] = -18[E_{i1}^u] * [E^\alpha] * [E_{1s}^u].$$

But by (6.1)-(6.2) and (6.15) we have

$$[E_{i1}^u] * [H_1] * [E^\alpha] * [E_{1s}^u] + [E_{i1}^u] * [E^\alpha] * [H_1] * [E_{1s}^u] = -9[E_{i1}^u] * [E^\alpha] * [E_{1s}^u].$$

This implies that $[E_{i1}^u] * [E^\alpha] * [E_{1s}^u] = 0$, as required. \square

We now define $E_{i,j}^u$ for all $i, j = 1, \dots, d+r+l$. Set

$$[E_{d+i,d+j}^u] = [E_{d+i,1}^u] * [E_{1,d+j}^u], i, j = 1, \dots, r+l.$$

It is easy to see that $[E_{d+i,1}^u] * [E_{1,d+j}^u] = [E_{d+i,k}^u] * [E_{k,d+j}^u]$, $k = 2, \dots, d$.

Denote by A_L^u the subalgebra of $A(V_L^+)$ generated by $\{[E_{ij}^u], [E_{d+p,j}^u], [E_{i,d+p}^u] | i, j = 1, \dots, d, p = 1, \dots, r+l\}$. From Lemma 6.4, (6.14) and the definition of $[E_{ij}^u]$, $i, j = 1, \dots, d+r+l$, we can easily deduce the following result.

Lemma 6.5. A_L^u is a matrix algebra over \mathbb{C} with basis $\{[E_{ij}^u] | i, j = 1, \dots, d+r+l\}$ such that

$$[E_{ij}^u] * [E_{ks}^u] = \delta_{j,k} [E_{is}^u], [E_{ij}^u] e^k = \delta_{j,k} e^i, i, j, k, s = 1, 2, \dots, d+r+l.$$

Lemma 6.6. Let $\alpha \in L$ and $\alpha \notin L_2$, then

$$[E^\alpha] * A^u = 0.$$

Proof: Let $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} such that $h_1 \in \mathbb{C}\alpha$. If $|\alpha|^2 = 2k$ and $k \neq 4$, the lemma follows from (6.1)-(6.2) and (6.13).

If $|\alpha|^2 = 8$, by (6.1)-(6.2) and (6.13) we have

$$[E_{ab}^u] * [E^\alpha] = [E^\alpha] * [E_{ba}^u] = 0,$$

for all $1 \leq a, b \leq d$ and $b \neq 1$. By (6.1) and (6.12), we have

$$[E_{a1}^u] * [H_1] * [E^\alpha] = 0. \quad (6.18)$$

On the other hand, for $a \neq 1$, by (6.7)-(6.8) and (6.3), we have

$$\begin{aligned} -\frac{2}{9}[E_{a1}^u] * ([H_a] - [H_1]) * [E^\alpha] &= -2[E_{a1}^u] * [E^\alpha], \\ \frac{4}{135}[E_{a1}^u] * (-13[H_a] + 15[H_1]) * [E^\alpha] &= -4[E_{a1}^u] * [E^\alpha]. \end{aligned}$$

Therefore by (6.18), we have

$$\begin{aligned}\frac{1}{9}[E_{a1}^u] * [H_a] * [E^\alpha] &= [E_{a1}^u] * [E^\alpha], \quad a \neq 1, \\ \frac{13}{135}[E_{a1}^u] * [H_a] * [E^\alpha] &= [E_{a1}^u] * [E^\alpha], \quad a \neq 1.\end{aligned}$$

This means that

$$[E_{a1}^u] * [E^\alpha] = 0.$$

Since $[H_\alpha] * [E^\alpha] = [E^\alpha] * [H_\alpha]$, we similarly have

$$[E^\alpha] * [E_{1a}^u] = 0.$$

This completes the proof. \square

Lemma 6.7. A_L^u is an ideal of $A(V_L^+)$.

Proof: By Proposition 5.4 of [AD], (6.3), (6.17) and Lemmas 6.4–6.6, it is enough to prove that $[E^{\alpha_i}] * [E_{jk}^u], [E_{jk}^u] * [E^{\alpha_i}] \in A_L^u$, $j, k = 1, \dots, d$, $i = 1, \dots, r + l$.

Let $\alpha \in L_2$. For convenience, let $a_{ij} = 0$, for $1 \leq i < j \leq r$ and $k_i = i$ for $1 \leq i \leq r$.

Since $\alpha_i(-1) = \sum_{k=1}^d a_{ik} h_k(-1)$, we have

$$\omega_{\alpha_i} = \frac{1}{4} \sum_{k=1}^d a_{ik}^2 h_k(-1)^2 + \frac{1}{4} \sum_{p \neq q} a_{ip} a_{iq} h_p(-1) h_q(-1).$$

Recall from [AD] that

$$[S_{ab}(1, 1)] = [E_{ab}^u] + [E_{ba}^u] + [\Lambda_{ab}] + \frac{1}{2}[E_{ab}^t] + \frac{1}{2}[E_{ba}^t], \quad a \neq b.$$

So from (6.3) and (6.4) we have

$$[E_{jk}^u] * [\omega_{\alpha_i}] = \frac{1}{2} a_{ik}^2 [E_{jk}^u] + \frac{1}{2} \sum_{p \neq k} a_{ik} a_{ip} [E_{jp}^u], \quad j, k = 1, \dots, d, \quad i = 1, \dots, r + l.$$

Then it can easily be deduced that

$$(a_{ik}[E_{jk_i}^u] - a_{ik_i}[E_{jk}^u]) * [\omega_{\alpha_i}] = 0, \quad j, k = 1, \dots, d, \quad i = 1, \dots, r + l.$$

Then by (6.16), we have

$$(a_{ik}[E_{jk_i}^u] - a_{ik_i}[E_{jk}^u]) * [E^{\alpha_i}] = 0, \quad j, k = 1, \dots, d, \quad i = 1, \dots, r + l.$$

So $[E_{jk}^u] * [E^{\alpha_i}] \in A_L^u$, for all $j, k = 1, \dots, d, i = 1, \dots, r + l$. Similarly, we have

$$[E^{\alpha_i}] * [E_{jk}^u] = 0, \quad k = 1, \dots, d, \quad i = 1, \dots, r + l, \quad j = r + 1, \dots, d,$$

$$[E^{\alpha_i}] * [\sum_{b=1}^r a_{kb} E_{bj}^u] = \frac{(\alpha_i, \alpha_k)}{2} [E^{\alpha_i}] * [\sum_{b=1}^r a_{ib} E_{bj}^u], \quad j = 1, \dots, d, \quad k = 1, \dots, r, \quad i = 1, \dots, r+l.$$

Since both $\{\alpha_1, \dots, \alpha_r\}$ and $\{h_1, \dots, h_r\}$ are linearly independent, it follows that for each $i = 1, \dots, r$, $j = 1, \dots, d$, $[E_{ij}^u]$ is a linear combination of $a_{11}[E_{1j}^u]$, $[a_{21}E_{1j}^u + a_{22}E_{2j}^u], \dots, [a_{r1}E_{1j}^u + \dots + a_{rr}E_{rj}^u]$. Therefore $[E^{\alpha_i}] * [E_{jk}^u] \in A_L^u$, $j, k = 1, \dots, d$, $i = 1, \dots, r+l$. \square

For $0 \neq \alpha \in L$, let $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} such that $h_1 \in \mathbb{C}\alpha$. Define

$$[B_\alpha] = 2^{|\alpha|^2-1}([I^t] * [E^\alpha] - \frac{2|\alpha|^2}{2|\alpha|^2-1}[E_{11}^t] * [E^\alpha]),$$

and $[B_0] = [I^t]$ (see formula (6.5) of [AD]).

Lemma 6.8. For $\alpha \in L$, $[E_{ij}^t] \in A^t$, $[B_\alpha] * [E_{ij}^t] = [E_{ij}^t] * [B_\alpha]$.

Proof: It is enough to prove that

$$[B_\alpha] * [E_{ij}^t] = [E_{ij}^t] * [B_\alpha],$$

for $i = 1$ or $j = 1$. By the definition of $[E_{ab}^t]$ and the fact that

$$[I^t] * [E^\alpha] = [E^\alpha] * [I^t]$$

and

$$[I^t] * [\Lambda_{ab}] = [I^t] * [E_{ab}^u] = 0, \quad a \neq b,$$

we have

$$\begin{aligned} [B_\alpha] * [E_{ab}^t] &= [B_\alpha] * (-[S_{ab}(1, 1)] - 2[S_{ab}(1, 2)]), \\ [E_{ab}^t] * [B_\alpha] &= (-[S_{ab}(1, 1)] - 2[S_{ab}(1, 2)]) * [B_\alpha]. \end{aligned}$$

Let $b \neq 1$. Similar to the proof of Lemma 7.5 of [AD], we have

$$\begin{aligned} & (2|\alpha|^2 - 1)([E_{1b}^t] + 3[E_{1b}^u] + [\Lambda_{1b}]) * [E^\alpha] + [E^\alpha] * ([E_{1b}^t] + 3[E_{1b}^u] + [\Lambda_{1b}]) \\ &= -([E_{b1}^t] - [E_{1b}^u] + [\Lambda_{1b}]) * [E^\alpha] - (2|\alpha|^2 - 1)[E^\alpha] * ([E_{b1}^t] - [E_{1b}^u] + [\Lambda_{1b}]) \\ &= (2|\alpha|^2 - 1)(\frac{1}{16}[E_{1b}^t] * [E^\alpha] + [\omega_b] * [\Lambda_{1b}] * [E^\alpha]) + \frac{1}{16}[E^\alpha] * [E_{1b}^t] + [E^\alpha] * [\omega_b] * [\Lambda_{1b}] \\ &= -\frac{9}{16}[E_{b1}^t] * [E^\alpha] + [\omega_b] * [\Lambda_{1b}] * [E^\alpha] - (2|\alpha|^2 - 1)(\frac{9}{16}[E^\alpha] * [E_{b1}^t] + [E^\alpha] * [\omega_b] * [\Lambda_{1b}]). \end{aligned}$$

So we have

$$(2|\alpha|^2 - 1)[E_{1b}^t] * [E^\alpha] = -[E^\alpha] * [E_{1b}^t] + x,$$

where $x \in A_L^u + \mathbb{C}[\Lambda_{1b}] * [E^\alpha] + \mathbb{C}[E^\alpha] * [\Lambda_{1b}] + \mathbb{C}[\omega_b] * [\Lambda_{1b}] * [E^\alpha] + \mathbb{C}[E^\alpha] * [\Lambda_{1b}] * [\omega_b]$. Since $y * x = 0$ for any $y \in A^t$, we have

$$\begin{aligned} & [B_\alpha] * [E_{1b}^t] \\ &= 2^{|\alpha|^2-1}(-(2|\alpha|^2 - 1)[E_{1b}^t] * [E^\alpha] + 2|\alpha|^2[E_{11}^t] * [E_{1b}^t] * [E^\alpha]) \\ &= 2^{|\alpha|^2-1}[E_{1b}^t] * [E^\alpha] = [E_{1b}^t] * [B_\alpha]. \end{aligned}$$

Similarly,

$$[B_\alpha] * [E_{b1}^t] = [E_{b1}^t] * [B_\alpha],$$

completing the proof. \square

Lemma 6.9. A_L^t is an ideal of $A(V_L^+)$ and $A_L^t \cong A^t \otimes_{\mathbb{C}} \mathbb{C}[\widehat{L}/K]/J$, where $\mathbb{C}[\widehat{L}/K]$ is the group algebra of \widehat{L}/K and J is the ideal of $\mathbb{C}[\widehat{L}/K]$ generated by $\kappa K + 1$.

Proof: By Proposition 5.4 of [AD] and Lemmas 6.7-6.8, it is easy to check that A_L^t is an ideal of $A(V_L^+)$. Similar to the proof of Proposition 7.6 of [AD], we have

$$[B_\alpha] * [B_\beta] = \epsilon(\alpha, \beta)[B_{\alpha+\beta}],$$

for $\alpha, \beta \in L$ where $\epsilon(\alpha, \beta)$ is understood to be ± 1 by identifying κ with -1 . Then the lemma follows from Proposition 7.6 of [AD] and Lemma 6.8. \square

It is clear that $A_L^u \cap A_L^t = 0$. Let

$$\bar{A}(V_L^+) = A(V_L^+) / (A_L^u \oplus A_L^t),$$

and for $x \in A(V_L^+)$, we still denote the image of x in $\bar{A}(V_L^+)$ by x .

Lemma 6.10. In $\bar{A}(V_L^+)$, we have

$$[H_a] = [H_b], \quad 1 \leq a, b \leq d, \quad (6.19)$$

$$([\omega_a] - \frac{1}{16}) * [H_a] = 0, \quad 1 \leq a \leq d, \quad (6.20)$$

$$\frac{128}{9}[H_a] * \frac{128}{9}[H_a] = \frac{128}{9}[H_a], \quad 1 \leq a \leq d, \quad (6.21)$$

$$[\Lambda_{ab}] * [H_c] = 0, \quad 1 \leq a, b, c \leq d, \quad a \neq b. \quad (6.22)$$

Proof: (6.19) follows from (6.7) and (6.20) follows from (6.8) and (6.9). Then from (6.5) we can get (6.21). By (6.10), we have

$$[\Lambda_{ab}]^2 * [H_c] = 0, \quad a \neq b.$$

If $d \geq 3$, then by (6.19) we can let $c \neq a, c \neq b$. So by (6.11) and (6.20),

$$\begin{aligned} [\Lambda_{ab}] * [H_c] &= 16[\Lambda_{ab}] * [\omega_c] * [H_c] \\ &= 8[\Lambda_{ac}] * [\Lambda_{cb}] * [H_c] = 128[\Lambda_{ac}] * [\Lambda_{cb}] * [\omega_a] * [H_a] \\ &= 64[\Lambda_{ac}] * [\Lambda_{ca}] * [\Lambda_{ab}] * [H_a] = 0. \end{aligned}$$

If $d = 2$. Notice that $[\Lambda_{ab}] = [S_{ab}(1, 1)]$. By Remark 4.1.1 of [DN3] and the fact that $[\omega_a * S_{ab}(m, n)] = [S_{ab}(m, n) * \omega_a]$ in $\bar{A}(V_L^+)$ for $m, n \geq 1$, we have

$$[S_{ab}(m+1, n)] + [S_{ab}(m, n)] = 0. \quad (6.23)$$

By the proof of Lemma 6.1.2 of [DN3], we know that

$$[H_a] = -9[S_{aa}(1, 3)] - \frac{17}{2}[S_{aa}(1, 2)] + \frac{1}{2}[S_{aa}(1, 1)]. \quad (6.24)$$

Direct calculation yields

$$[S_{ab}(1, 1)] * [S_{aa}(1, 3)] = h_b(-1)h_a(-3)h_a(-1)^2,$$

$$[S_{ab}(1, 1)] * [S_{aa}(1, 2)] = h_b(-1)h_a(-2)h_a(-1)^2,$$

$$[S_{ab}(1, 1)] * [S_{aa}(1, 1)] = h_b(-1)h_a(-1)h_a(-1)^2.$$

Here we have used (6.23). Then (6.22) immediately follows from Lemma 4.2.1 of [DN3], (6.23) and (6.24). The proof is complete. \square

For $0 \neq \alpha \in L$, let $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} such that $h_1 \in \mathbb{C}\alpha$. Define

$$[\bar{B}_\alpha] = 2^{|\alpha|^2-1} \frac{128}{9} [H_1] * [E^\alpha].$$

We also set $[\bar{B}_0] = \frac{128}{9} [H_1]$.

Lemma 6.11. *The subalgebra A_H of $\bar{A}(V_L^+)$ spanned by $[\bar{B}_\alpha]$, $\alpha \in L$ is an ideal of $\bar{A}(V_L^+)$ isomorphic to $\mathbb{C}[\widehat{L}/K]/J$.*

Let

$$\widehat{A}(V_L^+) = \bar{A}(V_L^+)/A_H.$$

Lemma 6.12. *Any $\widehat{A}(V_L^+)$ -module is completely reducible. That is, $\widehat{A}(V_L^+)$ is a semisimple associative algebra.*

Proof: Let M be an $\widehat{A}(V_L^+)$ -module. For $\alpha \in L$, by [DN2] M is a direct sum of irreducible $A(V_{\mathbb{Z}\alpha}^+)$ -modules. Following the proof of Lemma 6.1 of [AD] one can prove that the image of any vector from $M(1)^+$ in $\widehat{A}(V_L^+)$ is semisimple on M . By Table 1 of [AD], we can assume that

$$M = \bigoplus_{\lambda \in \mathfrak{h}/(\pm 1)} M_\lambda,$$

where $M_\lambda = \{w \in M \mid [\frac{1}{2}h(-1)^2 1]w = \frac{1}{2}(\lambda, h)^2 w, h \in \mathfrak{h}\}$. So $\omega_a w = \frac{1}{2}(\lambda, h_a)^2 w$, for $w \in M_\lambda$. By (6.10) and (6.11), we have

$$\Lambda_{ab} w = (\lambda, h_a)(\lambda, h_b)w,$$

for $a \neq b$, $w \in M_\lambda$. For any $u \in M_\lambda$, $\lambda \neq 0$, set $M(u) = \sum_{\alpha \in L} \mathbb{C}[E^\alpha]u$. By (6.12)-(6.13) and (6.15)-(6.17), if $[E^\alpha]u \neq 0$, then $\alpha \in \Delta(\lambda)$ or $-\alpha \in \Delta(\lambda)$, where $\Delta(\lambda) = \{\alpha \in L \mid |\lambda + \alpha|^2 = |\lambda|^2\}$. So

$$M(u) = \bigoplus_{\alpha \in \Delta(\lambda)} \mathbb{C}[E^\alpha]u.$$

Since L is positive-definite, there are finitely many $\alpha \in L$ which belong to $\Delta(\lambda)$. Thus $M(u)$ is finite-dimensional. Similar to the proof of Lemma 6.4 of [AD], we can deduce that $\Lambda_{ab}M(u) \subseteq M(u)$, $\omega_a M(u) \subseteq M(u)$. By Proposition 5.4 of [AD], $[E^\alpha] * [E^\beta] = [x] * [E^{\alpha+\beta}]$ for some $x \in M(1)^+$. We deduce that $M(u)$ is an $\widehat{A}(V_L^+)$ -submodule of M . Suppose $[E^\alpha]u \neq 0$, for some $\alpha \in \Delta(\lambda)$. If $(\alpha, \alpha) = 2$, then by (6.14), we have $0 \neq [E^\alpha][E^\alpha]u \in \mathbb{C}u$. If $(\alpha, \alpha) = 2k \neq 2$. Let $\{h_1, \dots, h_d\}$ be an orthonormal basis of \mathfrak{h} such that $h_1 \in \mathbb{C}\alpha$. By the fact that $[H_1] = [J_1] + [\omega_1] - 4[\omega_1^2] = 0$ and (6.13) we know that $[\omega_1]u = \frac{k}{4}u$. Then by Lemma 5.5 of [DN2], we have

$$[E^\alpha][E^\alpha]u = \frac{2k^2}{(2k)!}(k^2 - 1)(k^2 - 2^2) \cdots (k^2 - (k-1)^2)u \neq 0.$$

Therefore $M(u)$ is irreducible. We prove that M is a direct sum of finite-dimensional irreducible $\widehat{A}(V_L^+)$ -module. \square

Theorem 6.13. V_L^+ is a rational vertex operator algebra.

Proof: By Lemmas 6.5-6.7, 6.9, 6.11 and 6.12, we know that $A(V_L^+)$ is semisimple as $\mathbb{C}[\widehat{L}/K]/J$ and $A^t \otimes \mathbb{C}[\widehat{L}/K]/J$ are semisimple. Then the theorem follows from Theorem 5.3. \square

References

- [A1] T. Abe, Rationality of the vertex operator algebra V_L^+ for a positive definite even lattice L , *Math. Z.* **249** (2005), 455-484.
- [A2] T. Abe, A Z_2 -orbifold model of the symplectic fermionic vertex operator superalgebra, math.QA/0503472.
- [ABD] T. Abe, G. Buhl and C. Dong, Rationality, regularity and C_2 -cofiniteness, *Trans. AMS.* **356** (2004), 3391-3402.
- [AD] T. Abe and C. Dong, Classification of irreducible modules for the vertex operator algebra V_L^+L : general case, *J. Algebra* **273** (2004), 657-685.
- [Ad] D. Adamović, Classification of irreducible modules of certain subalgebras of free boson vertex algebra, *J. Algebra* **270** (2003), 115-132.
- [B] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068-3071.
- [Bu] G. Buhl, A spanning set for VOA modules, *J. Algebra* **254** (2002), 125-151.
- [D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245-265.

- [D2] C. Dong, Representations of the moonshine module vertex operator algebra, *Contemp. Math.* **175** (1994), 27-36.
- [DGH] C. Dong, R. Griess Jr. and G. Hoehn, Framed vertex operator algebras, codes and the moonshine module, *Comm. Math. Phys.* **193** (1998), 407-448.
- [DJ1] C. Dong and C. Jiang, Bimodules associated to vertex operator algebras, math.QA/0601626.
- [DJ2] C. Dong and C. Jiang, Representation theory of vertex operator algebras *Contemp. Math.*, to appear, math.QA/0603588.
- [DLTY] C. Dong, C. Lam, K. Tanabe, H. Yamada and K. Yokoyama, \mathbb{Z}_3 symmetry and W_3 algebra in lattice vertex operator algebras, *Pacific J. Math.* **215** (2004), 245-296.
- [DL] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, *Progress in Math.* Vol. 112, Birkhäuser, Boston 1993.
- [DLM1] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, *Advances in Math.* **132** (1997), 148-166.
- [DLM2] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571-600.
- [DLM3] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67-96.
- [DLM4] C. Dong, H. Li and G. Mason, Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1-56.
- [DM1] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.*, **86** (1997), 305-321.
- [DM2] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *Internat. Math. Res. Notices* **56** (2004), 2989-3008.
- [DM3] C. Dong and G. Mason, Integrability of C_2 -cofinite vertex operator algebras, *Internat. Math. Res. Notices* **2006** (2006), Art. ID 80468, 15 pp.
- [DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math. American Math. Soc.* **56** II (1994), 295-316.
- [DN1] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$, *J. Algebra* **216** (1999), 384-404.

- [DN2] C. Dong and K. Nagatomo, Representations of vertex operator algebra V_L^+ for rank one lattice L , *Comm. Math. Phys.* **202** (1999), 169-195.
- [DN3] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$. II. Higher rank, *J. Algebra* **240** (2001), 289-325.
- [DZ] C. Dong and Z. Zhao, Twisted representations of vertex operator superalgebras, *Comm. Contemp. Math.* **8** (2006), 101-122.
- [FB] E. Frenkel and D. Ben-Ziv, Vertex Algebras and Algebraic Curves, *Mathematical Surveys and Monographs* **88**, AMS. 2001.
- [FHL] I. B. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **104**, 1993.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.* Vol. **134**, Academic Press, 1988.
- [FZ] I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J* **66** (1992), 123-168.
- [GN] M. Gaberdiel, and A. Neitzke, Rationality, quasirationality, and finite W-algebras, *Comm. Math. Phys.* **238** (2003), 305-331.
- [H] Y. Huang, Vertex operator algebras, the Verlinde conjecture, and modular tensor categories, *Proc. Natl. Acad. Sci. USA* **102** (2005), 5352-5356.
- [KR] V. Kac and A. Raina, Highest Weight Representations of Infinite Dimensional Lie Algebras, World Scientific, Adv. Ser. In Math. Phys., Singapore, 1987.
- [KW] V. Kac and W. Wang, Vertex operator superalgebras and their representations, *Contemp. Math. Amer. Math. Soc.* **175** (1994), 161-191.
- [KL] M. Karel and H. Li, Certain generating subspace for vertex operator algebras, *J. Algebra* **217** (1999), 393-421.
- [KMY] M. Kitazume, M. Miyamoto and H. Yamada, Ternary codes and vertex operator algebras, *J. Algebra* **223** (2000), 379-395.
- [LL] J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras and Their Representations, *Progress in Mathematics* Vol. **227**, Birkhuser Boston, Inc., Boston, MA, 2004.
- [L] H. Li, Some finiteness properties of regular vertex operator algebras, *J. Algebra* **212** (1999), 495-514.
- [M] M. Miyamoto, Representation theory of code vertex operator algebra, *J. Algebra* **201** (1998), 115-150.

- [NT] K. Nagatomo and A. Tsuchiya, Conformal field theories associated to regular chiral vertex operator algebras I, Theories over the projective line, *Duke Math. J.* **128** (2005), 393-471.
- [TY] K. Tanabe and H. Yamada, The fixed point subalgebra of a lattice vertex operator algebra by an automorphism of order three, math.QA/0508175.
- [W1] W. Wang, Rationality of Virasoro vertex operator algebras, *Internat. Math. Res. Notices* **7** (1993), 197-211.
- [W2] W. Wang, Classification of irreducible modules of W_3 algebra with $c = -2$, *Comm. Math. Phys.* **195** (1998), 113-128.
- [Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302.